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2008 J. Phys. A: Math. Theor. 41 475201

(http://iopscience.iop.org/1751-8121/41/47/475201)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 475201 (23pp) doi:10.1088/1751-8113/41/47/475201

A τ-function on the inhomogeneous lattice and the classical isotropic Heisenberg spin chain

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Received 24 June 2008, in final form 1 September 2008
Published 20 October 2008
Online at stacks.iop.org/JPhysA/41/475201

Abstract
A τ-function on the inhomogeneous lattice is introduced by a deformation of that of the Toda lattice hierarchy. An associated discrete-time integrable system is constructed in Sato and Lax formalisms. It is shown that a one-dimensional reduction of the system gives a Bäcklund transformation of the classical isotropic Heisenberg spin chain, which is equivalent to the R_{II}-chain proposed by Spiridonov and Zhedanov.

PACS numbers: 02.30.Ik, 05.45.Yv
Mathematics Subject Classification: 37K10

1. Introduction

The classical isotropic Heisenberg spin chain, introduced by Ishimori [3] and Sklyanin [13], is a model which describes a motion of ferromagnetic spins on the one-dimensional lattice. A characteristic of this model is a spatial inhomogeneity: the model contains two inhomogeneous parameters which depend on the lattice site. This inhomogeneity is closely related to a degeneration of the model. As pointed out by Sklyanin [14], the Heisenberg spin chain degenerates to the discrete self-trapping chain, which is also an inhomogeneous model with one parameter depending on the lattice site. The discrete self-trapping chain further degenerates to the Toda lattice, which has no nontrivial inhomogeneous parameter. All of the above models satisfy the same fundamental Poisson bracket. Therefore, the Heisenberg spin chain is considered as the most general model characterized by the fundamental Poisson bracket. Note that it was shown that the model in the homogeneous case is gauge equivalent to the spatially discrete nonlinear Schrödinger equation [3]. This fact indicates that the Heisenberg spin chain is essential in the inhomogeneous case.

From the viewpoint of the Sato theory, the Toda lattice is understood as a one-dimensional reduction of the Toda lattice hierarchy [20]. In this paper, we show that the Heisenberg spin chain is a one-dimensional reduction of an inhomogeneous deformation of the Toda lattice hierarchy. To be more precise, we introduce a τ-function on the inhomogeneous lattice by a deformation of that of the Toda lattice hierarchy and construct an associated discrete-time integrable system in Sato and Lax formalisms. Then we show that a one-dimensional reduction of the system gives a Bäcklund transformation of the Heisenberg spin chain, which turns out to be equivalent to the R_{II}-chain proposed by Spiridonov and Zhedanov [15, 16] in a study of a generalized eigenvalue problem.

This paper is organized as follows. In section 2, we consider an inhomogeneous deformation of the generalized vacuum vectors and discuss its relation to a variant of the Schur functions called the multiparameter Schur functions. In section 3, we introduce a τ-function on the inhomogeneous lattice and construct Sato equations. In section 4, we construct Lax equations and show that its one-dimensional reduction gives a Bäcklund transformation of the Heisenberg spin chain. Section 5 is devoted to concluding remarks.

2. Inhomogeneous lattice and multiparameter Schur functions

In this section we consider an inhomogeneous deformation of the generalized vacuum vectors and discuss its relation to a variant of the Schur functions called the multiparameter Schur functions [11].

We first recall the language of free fermions [4, 7]. Let $\psi_n, \psi_n^* (n \in \mathbb{Z})$ be operators satisfying the anticommutation relations

$$[\psi_m, \psi_n]_+ = [\psi_m^*, \psi_n^*]_+ = 0, \qquad [\psi_m, \psi_n^*]_+ = \delta_{mn}. \tag{2.1}$$

The operators ψ_n, ψ_n^* are called the free fermions. Define the vacuum vectors $\langle\mathrm{vac}|, |\mathrm{vac}\rangle$ by the properties

$$\begin{aligned} &\langle\mathrm{vac}|\psi_n = 0 \ (n \geqslant 0), \qquad \langle\mathrm{vac}|\psi_n^* = 0 \ (n < 0), \\ &\psi_n|\mathrm{vac}\rangle = 0 \ (n < 0), \qquad \psi_n^*|\mathrm{vac}\rangle = 0 \ (n \geqslant 0), \\ &\langle\mathrm{vac}|\mathrm{vac}\rangle = 1. \end{aligned} \tag{2.2}$$

We also define the generalized vacuum vectors $\langle l|, |l\rangle (l \in \mathbb{Z})$ by

$$\langle l| = \langle\mathrm{vac}|\Psi_l^*, \qquad |l\rangle = \Psi_l|\mathrm{vac}\rangle \tag{2.3}$$

with

$$\Psi_l^* = \begin{cases} \psi_{-1}\cdots\psi_l & (l < 0), \\ 1 & (l = 0), \\ \psi_0^*\cdots\psi_{l-1}^* & (l > 0), \end{cases} \qquad \Psi_l = \begin{cases} \psi_l^*\cdots\psi_{-1}^* & (l < 0), \\ 1 & (l = 0), \\ \psi_{l-1}\cdots\psi_0 & (l > 0). \end{cases} \tag{2.4}$$

The generalized vacuum vectors satisfy the orthogonality relation

$$\langle l|m\rangle = \delta_{lm}. \tag{2.5}$$

For pairs of infinite number of variables $x = (x_1, x_2, \ldots)$ and $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots)$, let us introduce the Hamiltonians as follows:

$$\begin{aligned} &H(x) = \sum_{n=1}^{\infty} x_n H_n, \qquad \bar{H}(\bar{x}) = \sum_{n=1}^{\infty} \bar{x}_n H_{-n}, \\ &H_n = \sum_{i\in\mathbb{Z}} :\psi_i\psi_{i+n}^*: \ (n \in \mathbb{Z}), \end{aligned} \tag{2.6}$$

where :: is the normal product

$$:\psi_m\psi_n^* := \psi_m\psi_n^* - \langle\mathrm{vac}|\psi_m\psi_n^*|\mathrm{vac}\rangle. \tag{2.7}$$

Then we can show that the free fermionic fields

$$\psi(z) = \sum_{n\in\mathbb{Z}} \psi_n z^n, \qquad \psi^*(z) = \sum_{n\in\mathbb{Z}} \psi_n^* z^{-n-1} \tag{2.8}$$

satisfy

$$\begin{aligned}
\mathrm{e}^{H(x)}\psi(z)\,\mathrm{e}^{-H(x)} &= \mathrm{e}^{\xi(x,z)}\psi(z),\\
\mathrm{e}^{H(x)}\psi^*(z)\,\mathrm{e}^{-H(x)} &= \mathrm{e}^{-\xi(x,z)}\psi^*(z),\\
\mathrm{e}^{\bar{H}(\bar{x})}\psi(z)\,\mathrm{e}^{-\bar{H}(\bar{x})} &= \mathrm{e}^{\xi(\bar{x},z^{-1})}\psi(z),\\
\mathrm{e}^{\bar{H}(\bar{x})}\psi^*(z)\,\mathrm{e}^{-\bar{H}(\bar{x})} &= \mathrm{e}^{-\xi(\bar{x},z^{-1})}\psi^*(z)
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
\langle l|\psi(z) &= z^{l-1}\langle l-1|\,\mathrm{e}^{H(-[z^{-1}])}, & \langle l|\psi^*(z) &= z^{-l-1}\langle l+1|\,\mathrm{e}^{H([z^{-1}])},\\
\psi(z)|l\rangle &= z^{l}\,\mathrm{e}^{-\bar{H}(-[z])}|l+1\rangle, & \psi^*(z)|l\rangle &= z^{-l}\,\mathrm{e}^{-\bar{H}([z])}|l-1\rangle.
\end{aligned} \tag{2.10}$$

Here we used the standard notation

$$\xi(x,z) = \sum_{n=1}^{\infty} x_n z^n, \qquad [z] = \left(z, \frac{z^2}{2}, \frac{z^3}{3}, \dots\right). \tag{2.11}$$

From (2.9) and (2.10) we obtain the so-called boson–fermion correspondence

$$\begin{aligned}
\langle l|\,\mathrm{e}^{H(x)}\psi(z) &= z^{l-1}X(x,z)\,\mathrm{e}^{-\partial_l}\langle l|\,\mathrm{e}^{H(x)},\\
\langle l|\,\mathrm{e}^{H(x)}\psi^*(z) &= z^{-l-1}X^*(x,z)\,\mathrm{e}^{\partial_l}\langle l|\,\mathrm{e}^{H(x)},\\
\psi(z)\,\mathrm{e}^{-\bar{H}(\bar{x})}|l\rangle &= z^{l}X(\bar{x},z^{-1})\,\mathrm{e}^{\partial_l}\,\mathrm{e}^{-\bar{H}(\bar{x})}|l\rangle,\\
\psi^*(z)\,\mathrm{e}^{-\bar{H}(\bar{x})}|l\rangle &= z^{-l}X^*(\bar{x},z^{-1})\,\mathrm{e}^{-\partial_l}\,\mathrm{e}^{-\bar{H}(\bar{x})}|l\rangle,
\end{aligned} \tag{2.12}$$

where $X(x,z)$, $X^*(x,z)$ are the vertex operators

$$\begin{aligned}
&X(x,z) = \mathrm{e}^{\xi(x,z)}\,\mathrm{e}^{-\xi(\tilde{\partial}_x,z^{-1})}, \qquad X^*(x,z) = \mathrm{e}^{-\xi(x,z)}\,\mathrm{e}^{\xi(\tilde{\partial}_x,z^{-1})},\\
&\tilde{\partial}_x = \left(\frac{\partial}{\partial x_1}, \frac{1}{2}\frac{\partial}{\partial x_2}, \frac{1}{3}\frac{\partial}{\partial x_3}, \dots\right).
\end{aligned} \tag{2.13}$$

Let $a = (a_l)_{l\in\mathbb{Z}}$ be a sequence mutually distinct in $\mathbb{C}\backslash\{0\}$, and set $a^{-1} = \left(a_l^{-1}\right)_{l\in\mathbb{Z}}$, $\bar{a} = (a_{-l-1})_{l\in\mathbb{Z}}$. We consider a deformation of the generalized vacuum vectors $\langle l(a)|$, $|l(a)\rangle$ defined by

$$\begin{aligned}
\langle l(a)| &= \begin{cases} \langle l|\,\mathrm{e}^{H([a_0^{-1}]+\cdots+[a_{l-1}^{-1}])} & (l>0),\\ \langle \mathrm{vac}| & (l=0),\\ \langle l|\,\mathrm{e}^{H(-[a_{-1}^{-1}]-\cdots-[a_l^{-1}])} & (l<0), \end{cases}\\
|l(a)\rangle &= \begin{cases} \mathrm{e}^{-\bar{H}(-[a_0]-\cdots-[a_{l-1}])}|l\rangle & (l>0),\\ |\mathrm{vac}\rangle & (l=0),\\ \mathrm{e}^{-\bar{H}([a_{-1}]+\cdots+[a_l])}|l\rangle & (l<0). \end{cases}
\end{aligned} \tag{2.14}$$

If we set

$$\begin{aligned}
\Psi_l^*(a) &= \begin{cases} \psi(a_{-1})\cdots\psi(a_l) & (l<0),\\ 1 & (l=0),\\ \psi^*(a_0)\cdots\psi^*(a_{l-1}) & (l>0), \end{cases}\\
\Psi_l(a) &= \begin{cases} \psi^*(a_l)\cdots\psi^*(a_{-1}) & (l<0),\\ 1 & (l=0),\\ \psi(a_{l-1})\cdots\psi(a_0) & (l>0), \end{cases}
\end{aligned} \tag{2.15}$$

then we have the expressions analogous to (2.3),

$$\begin{aligned}
\langle l(a)| &= \langle \mathrm{vac}|\Psi_l^*(a)\Psi_l|\mathrm{vac}\rangle^{-1}\langle \mathrm{vac}|\Psi_l^*(a),\\
|l(a)\rangle &= \langle \mathrm{vac}|\Psi_l^*\Psi_l(a)|\mathrm{vac}\rangle^{-1}\Psi_l(a)|\mathrm{vac}\rangle.
\end{aligned} \tag{2.16}$$

The orthogonality relation (2.5) is modified as

$$\begin{aligned} &\langle l(a)|m\rangle = \delta_{lm}, \qquad \langle l|m(a)\rangle = \delta_{lm},\\ &\langle l(a)|m(b)\rangle = \frac{\langle \mathrm{vac}|\Psi_l^*(a)\Psi_l(b)|\mathrm{vac}\rangle}{\langle \mathrm{vac}|\Psi_l^*(a)\Psi_l|\mathrm{vac}\rangle\langle \mathrm{vac}|\Psi_l^*\Psi_l(b)|\mathrm{vac}\rangle}\delta_{lm}. \end{aligned} \tag{2.17}$$

We call the deformed generalized vacuum vectors $\langle l(a)|$, $|l(a)\rangle$ the inhomogeneous lattice and the original ones $\langle l|$, $|l\rangle$ the homogeneous lattice. Using (2.10) recursively we have

$$\begin{aligned} &\langle l(a)|\psi(z) = \frac{1}{(z^{-1}|a^{-1})_{l-1}}\,\mathrm{e}^{\xi([a_{l-1}^{-1}],z)}\langle l-1(a)|\,\mathrm{e}^{H([a_{l-1}^{-1}]-[z^{-1}])},\\ &\langle l(a)|\psi^*(z) = (z^{-1}|a^{-1})_{l+1}\,\mathrm{e}^{\xi(-[a_l^{-1}],z)}\langle l+1(a)|\,\mathrm{e}^{H(-[a_l^{-1}]+[z^{-1}])},\\ &\psi(z)|l(a)\rangle = (z|a)_l\,\mathrm{e}^{-\bar H([a_l]-[z])}|l+1(a)\rangle,\\ &\psi^*(z)|l(a)\rangle = \frac{1}{(z|a)_l}\,\mathrm{e}^{-\bar H(-[a_{l-1}]+[z])}|l-1(a)\rangle, \end{aligned} \tag{2.18}$$

where we introduced the generalized shifted factorial

$$(z|a)_l = \begin{cases} (z-a_0)\cdots(z-a_{l-1}) & (l>0),\\ 1 & (l=0),\\ (z-a_{-1})^{-1}\cdots(z-a_l)^{-1} & (l<0). \end{cases} \tag{2.19}$$

From (2.9) and (2.18) we obtain a counterpart of (2.12)

$$\begin{aligned} &\langle l(a)|\,\mathrm{e}^{H(x)}\psi(z) = \frac{1}{(z^{-1}|a^{-1})_{l-1}}\,\mathrm{e}^{\xi(\tilde\partial_x,a_{l-1}^{-1})}X(x,z)\,\mathrm{e}^{-\partial_l}\langle l(a)|\,\mathrm{e}^{H(x)},\\ &\langle l(a)|\,\mathrm{e}^{H(x)}\psi^*(z) = (z^{-1}|a^{-1})_{l+1}\,\mathrm{e}^{-\xi(\tilde\partial_x,a_l^{-1})}X^*(x,z)\,\mathrm{e}^{\partial_l}\langle l(a)|\,\mathrm{e}^{H(x)},\\ &\psi(z)\,\mathrm{e}^{-\bar H(\bar x)}|l(a)\rangle = (z|a)_l X(\bar x,z^{-1})\,\mathrm{e}^{\xi(\tilde\partial_{\bar x},a_l)}\,\mathrm{e}^{\partial_l}\,\mathrm{e}^{-\bar H(\bar x)}|l(a)\rangle,\\ &\psi^*(z)\,\mathrm{e}^{-\bar H(\bar x)}|l(a)\rangle = \frac{1}{(z|a)_l}X^*(\bar x,z^{-1})\,\mathrm{e}^{-\xi(\tilde\partial_{\bar x},a_{l-1})}\,\mathrm{e}^{-\partial_l}\,\mathrm{e}^{-\bar H(\bar x)}|l(a)\rangle. \end{aligned} \tag{2.20}$$

It is well known that the Schur functions are realized by the action of the vertex operators on the constant function [4]. We next discuss a relation between the vertex operators on the inhomogeneous lattice and a variant of the Schur functions called the multiparameter Schur functions [11]. We first consider a shift operator $T_{x,z}^1(a)$ defined by

$$T_{x,z}^1(a) = \mathrm{e}^{\xi(\tilde\partial_x,a_0)}\,\mathrm{e}^{-\xi(\tilde\partial_x,z)}, \tag{2.21}$$

which is the annihilation part of the vertex operator $X(x,z^{-1})\,\mathrm{e}^{\xi(\tilde\partial_x,a_0)}$. Formal expansion of the operator generates difference operators $\Delta_x^l(a)(l \geqslant 0)$,

$$T_{x,z}^1(a) = \sum_{l=0}^{\infty}\Delta_x^l(a)(z|a)_l, \tag{2.22}$$

for example,

$$\begin{aligned} &\Delta_x^0(a) = 1,\\ &\Delta_x^1(a) = \frac{1-\mathrm{e}^{\xi(\tilde\partial_x,a_0)}\,\mathrm{e}^{-\xi(\tilde\partial_x,a_1)}}{a_0-a_1},\\ &\Delta_x^2(a) = \frac{1}{a_1-a_2}\left(\frac{1-\mathrm{e}^{\xi(\tilde\partial_x,a_0)}\,\mathrm{e}^{-\xi(\tilde\partial_x,a_1)}}{a_0-a_1} - \frac{1-\mathrm{e}^{\xi(\tilde\partial_x,a_0)}\,\mathrm{e}^{-\xi(\tilde\partial_x,a_2)}}{a_0-a_2}\right). \end{aligned} \tag{2.23}$$

Define a difference operator $D^1_{x,z}(a)$ by

$$D^1_{x,z}(a) = \frac{T^1_{x,z}(a) - 1}{a_0 - z}. \tag{2.24}$$

The difference operator acts on the exponential function as

$$D^1_{x,\lambda}(a)\,\mathrm{e}^{\xi(x,z^{-1})} = \frac{1}{z - a_0}\,\mathrm{e}^{\xi(x,z^{-1})}. \tag{2.25}$$

If we introduce higher order shift and difference operators as

$$\begin{aligned} T^n_{x,z}(a) &= \frac{T^{n-1}_{x,z}(a) - T^{n-1}_{x,z}(\tau a)}{a_0 - a_{n-1}}, \\ D^n_{x,z}(a) &= \frac{D^{n-1}_{x,z}(a) - D^{n-1}_{x,z}(\tau a)}{a_0 - a_{n-1}} = \frac{T^n_{x,z}(a) - D^{n-1}_{x,z}(a)}{a_{n-1} - z}, \end{aligned} \tag{2.26}$$

then we have

$$D^n_{x,\lambda}(a)\,\mathrm{e}^{\xi(x,z^{-1})} = \frac{1}{(z|a)_n}\,\mathrm{e}^{\xi(x,z^{-1})}. \tag{2.27}$$

Next we consider the creation part of the vertex operator. Let τ be an operator acting on $a = (a_l)_{l\in\mathbb{Z}}$ as $\tau a = (a_{l+1})_{l\in\mathbb{Z}}$. The multiparameter Schur function [11] indexed to a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ is defined by

$$S_\lambda(x|a) = \det\left(p_{\lambda_i - i + j}(x|\tau^{1-j}a)\right)_{1\leqslant i,j\leqslant N}. \tag{2.28}$$

Here $p_l(x|a) (l \in \mathbb{Z})$ are defined by the generating function

$$\mathrm{e}^{\xi(x,z^{-1})} = \sum_{l=0}^{\infty} p_l(x|a)\frac{1}{(z|a)_l} \tag{2.29}$$

and $p_l(x|a) = 0 (l < 0)$, for example,

$$\begin{aligned} p_0(x|a) &= 1, \\ p_1(x|a) &= x_1, \\ p_2(x|a) &= \left(x_2 + \frac{x_1^2}{2}\right) - a_0 x_1, \\ p_3(x|a) &= \left(x_3 + x_1 x_2 + \frac{x_1^3}{3}\right) - (a_0 + a_1)\left(x_2 + \frac{x_1^2}{2}\right) + a_0 a_1 x_1. \end{aligned} \tag{2.30}$$

The following basic properties can be easily shown:

$$\begin{aligned} &p_l(0|a) = \delta_{l0}, \\ &\frac{\partial}{\partial x_n}(1 + a_{n-1}\,\mathrm{e}^{-\partial_l}\tau)\cdots(1 + a_0\,\mathrm{e}^{-\partial_l}\tau)p_l(x|a) = p_{l-n}(x|\tau^n a), \\ &p_l(x|\tau a) = p_l(x|a) + (a_0 - a_{l-1})p_{l-1}(x|a). \end{aligned} \tag{2.31}$$

Especially we note

$$\Delta^n_x(a)p_l(x|a) = p_l(x|a)\delta_{n0} - p_{l-1}(x|\tau a)\delta_{n1}. \tag{2.32}$$

As in the case of the Schur functions, the multiparameter Schur functions have the following generating function formula:

$$\begin{aligned} &\sum_{\lambda_1=-N+1}^{\infty}\cdots\sum_{\lambda_N=0}^{\infty} S_{(\lambda_1,\lambda_2,\ldots,\lambda_N)}(x|a)\frac{1}{(z_1|a)_{\lambda_1}}\cdots\frac{1}{(z_N|\tau^{-N+1}a)_{\lambda_N}} \\ &\quad = X\left(x, z_1^{-1}\right)\mathrm{e}^{\xi(\tilde{\partial}_x, a_{-1})}\cdots X\left(x, z_N^{-1}\right)\mathrm{e}^{\xi(\tilde{\partial}_x, a_{-N})}\cdot 1, \end{aligned} \tag{2.33}$$

where the operators on the right-hand side are the vertex operators on the inhomogeneous lattice. We show this formula. By definitions (2.28) and (2.29), one has

$$\begin{aligned}
&\sum_{\lambda_1=-N+1}^{\infty}\cdots\sum_{\lambda_N=0}^{\infty} S_{(\lambda_1,\lambda_2,\ldots,\lambda_N)}(x|\tau^{N-1}a)\frac{1}{(z_1|a)_{\lambda_1+N-1}}\cdots\frac{1}{(z_N|a)_{\lambda_N}}\\
&\quad=\det\left(\sum_{\lambda_i=-N+i}^{\infty} p_{\lambda_i-i+j}(x|\tau^{N-j}a)\frac{1}{(z_i|a)_{\lambda_i+N-i}}\right)_{1\leqslant i,j\leqslant N}\\
&\quad=\det\left(\frac{1}{(z_i|a)_{N-j}}\right)_{1\leqslant i,j\leqslant N} \mathrm{e}^{\xi(x,z_1^{-1})}\cdots\mathrm{e}^{\xi(x,z_N^{-1})}\\
&\quad=\frac{\prod_{i>j}(z_i-z_j)}{\prod_{i=1}^{N}(z_i|a)_{N-1}}\,\mathrm{e}^{\xi(x,z_1^{-1})}\cdots\mathrm{e}^{\xi(x,z_N^{-1})}.
\end{aligned}\tag{2.34}$$

Multiplying the both hand sides by $(z_1|a)_{N-1}(z_2|a)_{N-2}\cdots 1$,

$$\begin{aligned}
&\sum_{\lambda_1=-N+1}^{\infty}\cdots\sum_{\lambda_N=0}^{\infty} S_{(\lambda_1,\lambda_2,\ldots,\lambda_N)}(x|\tau^{N-1}a)\frac{1}{(z_1|\tau^{N-1}a)_{\lambda_1}}\cdots\frac{1}{(z_N|a)_{\lambda_N}}\\
&\quad=\prod_{i>j}\frac{z_i-z_j}{z_i-a_{N-j-1}}\,\mathrm{e}^{\xi(x,z_1^{-1})}\cdots\mathrm{e}^{\xi(x,z_N^{-1})}\\
&\quad=\prod_{i>j}\frac{z_i-z_j}{z_i-a_{N-j-1}}\,\mathrm{e}^{\xi(x,z_1^{-1})}\cdots\mathrm{e}^{\xi(x,z_N^{-1})}\,\mathrm{e}^{-\xi(\tilde{\partial}_x,z_1)}\cdots\mathrm{e}^{-\xi(\tilde{\partial}_x,z_N)}\\
&\qquad\cdot\mathrm{e}^{\xi(\tilde{\partial}_x,a_{N-2})}\cdots\mathrm{e}^{\xi(\tilde{\partial}_x,a_{-1})}\cdot 1.
\end{aligned}\tag{2.35}$$

Reordering the operators on the right-hand side and operating τ^{-N+1} on the both hand sides, one obtains (2.33). This formula leads to an expression of the multiparameter Schur functions

$$S_{(\lambda_1,\lambda_2,\ldots,\lambda_N)}(x|a)=X_{\lambda_1}(x|a)X_{\lambda_2}(x|\tau^{-1}a)\cdots X_{\lambda_N}(x|\tau^{-N+1}a)\cdot 1,\tag{2.36}$$

where $X_l(x|a)(l\in\mathbb{Z})$ are the operators defined by

$$X(x,z^{-1})\,\mathrm{e}^{\xi(\tilde{\partial}_x,a_{-1})}=\sum_{l\in\mathbb{Z}}X_l(x|a)\frac{1}{(z|a)_l},\tag{2.37}$$

for example,

$$\begin{aligned}
X_{-1}(x|a)&=p_0(x|\tau^{-1}a)\Delta_x^1(\bar{a})+p_1(x|\tau^{-2}a)\Delta_x^2(\bar{a})+p_2(x|\tau^{-3}a)\Delta_x^3(\bar{a})+\cdots,\\
X_0(x|a)&=p_0(x|a)\Delta_x^0(\bar{a})+p_1(x|\tau^{-1}a)\Delta_x^1(\bar{a})+p_2(x|\tau^{-2}a)\Delta_x^2(\bar{a})+\cdots,\\
X_1(x|a)&=p_1(x|a)\Delta_x^0(\bar{a})+p_2(x|\tau^{-1}a)\Delta_x^1(\bar{a})+p_3(x|\tau^{-2}a)\Delta_x^2(\bar{a})+\cdots.
\end{aligned}\tag{2.38}$$

3. τ-function on inhomogeneous lattice and Sato equations

In this section we introduce a τ-function on the inhomogeneous lattice and construct Sato equations.

The quadratic expressions of free fermions constitute the affine Lie algebra $\mathfrak{gl}(\infty)$. For any element of the corresponding group, $g\in GL(\infty)$, we introduce a Toda lattice type

τ-function on the inhomogeneous lattice by

$$\tau(l, x, \bar{x}) = \tau(l, x, \bar{x}|a, b; g) = \langle l(a)|\, \mathrm{e}^{H(x)} g\, \mathrm{e}^{-\bar{H}(\bar{x})} |l(b)\rangle. \tag{3.1}$$

In the homogeneous limit $a_l \to \infty$, $b_l \to 0$, the τ-function reduces to that of the Toda lattice hierarchy [18]. The τ-function for $g = 1$ is given by

$$\tau(l, x, \bar{x}|a, b; 1) = \begin{cases} \mathrm{e}^{-\sum_{n=1}^{\infty} n x_n \bar{x}_n}\, \mathrm{e}^{\sum_{i=0}^{l-1} \xi(x, b_i)}\, \mathrm{e}^{-\sum_{i=0}^{l-1} \xi(\bar{x}, a_i^{-1})} \langle l(a)|l(b)\rangle & (l > 0), \\ \mathrm{e}^{-\sum_{n=1}^{\infty} n x_n \bar{x}_n} & (l = 0), \\ \mathrm{e}^{-\sum_{n=1}^{\infty} n x_n \bar{x}_n}\, \mathrm{e}^{-\sum_{i=1}^{-l} \xi(x, b_{-i})}\, \mathrm{e}^{\sum_{i=1}^{-l} \xi(\bar{x}, a_{-i}^{-1})} \langle l(a)|l(b)\rangle & (l < 0). \end{cases} \tag{3.2}$$

As in the case of the Toda lattice hierarchy [20], the τ-function (3.1) satisfies the bilinear identity

$$\begin{aligned} &\oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{z^{-2}(z^{-1}|a^{-1})_{l'-1}}{(z^{-1}|a^{-1})_{l+1}}\, \mathrm{e}^{\xi(x-x', z)} \tau\left(l, x + \left[a_l^{-1}\right] - [z^{-1}], \bar{x}\right) \tau\left(l', x' - \left[a_{l'-1}^{-1}\right] + [z^{-1}], \bar{x}'\right) \\ &\quad = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{(z|b)_l}{(z|b)_{l'}}\, \mathrm{e}^{\xi(\bar{x}-\bar{x}', z^{-1})} \tau(l+1, x, \bar{x} + [b_l] - [z]) \\ &\quad \times \tau(l'-1, x', \bar{x}' - [b_{l'-1}] + [z]), \end{aligned} \tag{3.3}$$

where the integrals in the both hand sides are understood as the operation for formal Laurent series,

$$\oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} z^n = \delta_{n,-1}. \tag{3.4}$$

Introduce wavefunctions $w(l, x, \bar{x}, z)$, $\bar{w}(l, x, \bar{x}, z)$ and their adjoints $w^*(l, x, \bar{x}, z)$, $\bar{w}^*(l, x, \bar{x}, z)$ as follows:

$$\begin{aligned} w(l, x, \bar{x}, z) &= \frac{\langle l+1(a)|\, \mathrm{e}^{H(x)} \psi(z) g\, \mathrm{e}^{-\bar{H}(\bar{x})} |l(b)\rangle}{\langle l(a)|\, \mathrm{e}^{H(x+[a_l^{-1}])} g\, \mathrm{e}^{-\bar{H}(\bar{x})} |l(b)\rangle}, \\ w^*(l, x, \bar{x}, z) &= \frac{\langle l-1(a)|\, \mathrm{e}^{H(x)} \psi^*(z) g\, \mathrm{e}^{-\bar{H}(\bar{x})} |l(b)\rangle}{\langle l(a)|\, \mathrm{e}^{H(x-[a_{l-1}^{-1}])} g\, \mathrm{e}^{-\bar{H}(\bar{x})} |l(b)\rangle}, \\ \bar{w}(l, x, \bar{x}, z) &= \frac{\langle l+1(a)|\, \mathrm{e}^{H(x)} g \psi(z)\, \mathrm{e}^{-\bar{H}(\bar{x})} |l(b)\rangle}{\langle l(a)|\, \mathrm{e}^{H(x+[a_l^{-1}])} g\, \mathrm{e}^{-\bar{H}(\bar{x})} |l(b)\rangle}, \\ \bar{w}^*(l, x, \bar{x}, z) &= \frac{\langle l-1(a)|\, \mathrm{e}^{H(x)} g \psi^*(z)\, \mathrm{e}^{-\bar{H}(\bar{x})} |l(b)\rangle}{\langle l(a)|\, \mathrm{e}^{H(x-[a_{l-1}^{-1}])} g\, \mathrm{e}^{-\bar{H}(\bar{x})} |l(b)\rangle}. \end{aligned} \tag{3.5}$$

Using (2.20) we can express the wavefunctions in ratios of the τ-function

$$\begin{aligned} w(l, x, \bar{x}, z) &= \frac{z^{-1}}{(z^{-1}|a^{-1})_{l+1}}\, \mathrm{e}^{\xi(x,z)} \frac{\tau\left(l, x + \left[a_l^{-1}\right] - [z^{-1}], \bar{x}\right)}{\tau\left(l, x + \left[a_l^{-1}\right], \bar{x}\right)}, \\ w^*(l, x, \bar{x}, z) &= z^{-1}(z^{-1}|a^{-1})_{l-1}\, \mathrm{e}^{-\xi(x,z)} \frac{\tau\left(l, x - \left[a_{l-1}^{-1}\right] + [z^{-1}], \bar{x}\right)}{\tau\left(l, x - \left[a_{l-1}^{-1}\right], \bar{x}\right)}, \\ \bar{w}(l, x, \bar{x}, z) &= (z|b)_l\, \mathrm{e}^{\xi(\bar{x}, z^{-1})} \frac{\tau(l+1, x, \bar{x} + [b_l] - [z])}{\tau\left(l, x + \left[a_l^{-1}\right], \bar{x}\right)}, \\ \bar{w}^*(l, x, \bar{x}, z) &= \frac{1}{(z|b)_l}\, \mathrm{e}^{-\xi(\bar{x}, z^{-1})} \frac{\tau(l-1, x, \bar{x} - [b_{l-1}] + [z])}{\tau\left(l, x - \left[a_{l-1}^{-1}\right], \bar{x}\right)}. \end{aligned} \tag{3.6}$$

Thus the bilinear identity (3.3) is rewritten as

$$\oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} w(l,x,\bar{x},z) w^*(l',x',\bar{x}',z) = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \bar{w}(l,x,\bar{x},z)\bar{w}^*(l',x',\bar{x}',z). \tag{3.7}$$

We define Sato–Wilson operators by

$$\begin{aligned} W(l,x,\bar{x}) &= \sum_{n=0}^{\infty} w_n(l,x,\bar{x})\,\mathrm{e}^{-n\partial_l}, \qquad W^*(l,x,\bar{x}) = \sum_{n=0}^{\infty} w_n^*(l+1-n,x,\bar{x})\,\mathrm{e}^{-n\partial_l}, \\ \bar{W}(l,x,\bar{x}) &= \sum_{n=0}^{\infty} \bar{w}_n(l,x,\bar{x})\,\mathrm{e}^{n\partial_l}, \qquad \bar{W}^*(l,x,\bar{x}) = \sum_{n=0}^{\infty} \bar{w}_n^*(l+1+n,x,\bar{x})\,\mathrm{e}^{n\partial_l}, \end{aligned} \tag{3.8}$$

where $w_n(l,x,\bar{x})$, $w_n^*(l,x,\bar{x})$, $\bar{w}_n(l,x,\bar{x})$ and $\bar{w}_n^*(l,x,\bar{x})$ denote the coefficients of the formal expansions

$$\begin{aligned} w(l,x,\bar{x},z) &= \mathrm{e}^{\xi(x,z)} \sum_{n=0}^{\infty} w_n(l,x,\bar{x}) \frac{z^{-1}}{(z^{-1}|a^{-1})_{l+1-n}}, \\ w^*(l,x,\bar{x},z) &= \mathrm{e}^{-\xi(x,z)} \sum_{n=0}^{\infty} w_n^*(l,x,\bar{x}) z^{-1}(z^{-1}|a^{-1})_{l-1+n}, \\ \bar{w}(l,x,\bar{x},z) &= \mathrm{e}^{\xi(\bar{x},z^{-1})} \sum_{n=0}^{\infty} \bar{w}_n(l,x,\bar{x})(z|b)_{l+n}, \\ \bar{w}^*(l,x,\bar{x},z) &= \mathrm{e}^{-\xi(\bar{x},z^{-1})} \sum_{n=0}^{\infty} \bar{w}_n^*(l,x,\bar{x}) \frac{1}{(z|b)_{l-n}}. \end{aligned} \tag{3.9}$$

The wavefunctions are expressed in terms of the Sato–Wilson operators as follows:

$$\begin{aligned} w(l,x,\bar{x},z) &= W(l,x,\bar{x})\big(1 + a_l^{-1}\,\mathrm{e}^{\partial_l}\big)\frac{1}{(z^{-1}|a^{-1})_l}\,\mathrm{e}^{\xi(x,z)}, \\ \bar{w}(l,x,\bar{x},z) &= \bar{W}(l,x,\bar{x})(z|b)_l\,\mathrm{e}^{\xi(\bar{x},z^{-1})}. \end{aligned} \tag{3.10}$$

Here we note that the following relation holds:

$$\oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{(z|a)_l}{(z|a)_{l'}} z^n = (a_l + \mathrm{e}^{\partial_l})^n \delta_{ll'-1} = (a_{l'-1} + \mathrm{e}^{-\partial_{l'}})^n \delta_{ll'-1}. \tag{3.11}$$

For negative n the operators $(a_l + \mathrm{e}^{\partial_l})^n$ and $(a_{l'-1} + \mathrm{e}^{-\partial_{l'}})^n$ are understood as the nth powers of

$$\begin{aligned} (a_l + \mathrm{e}^{\partial_l})^{-1} &= \mathrm{e}^{-\partial_l} - a_{l-1}\,\mathrm{e}^{-2\partial_l} + a_{l-1}a_{l-2}\,\mathrm{e}^{-3\partial_l} - \cdots, \\ (a_{l'-1} + \mathrm{e}^{-\partial_{l'}})^{-1} &= \mathrm{e}^{\partial_{l'}} - a_{l'}\,\mathrm{e}^{2\partial_{l'}} + a_{l'}a_{l'+1}\,\mathrm{e}^{3\partial_{l'}} + \cdots, \end{aligned} \tag{3.12}$$

respectively. Then from bilinear identity (3.7), we can obtain a relation among the Sato–Wilson operators

$$\begin{aligned} &W(l,x,\bar{x})\,\mathrm{e}^{\xi(x-x',(a_l^{-1}+\mathrm{e}^{-\partial_l})^{-1})}\,W(l,x',\bar{x}')^{-1} \\ &\quad = \bar{W}(l,x,\bar{x})\,\mathrm{e}^{\xi(\bar{x}-\bar{x}',(b_l+\mathrm{e}^{\partial_l})^{-1})}\,\bar{W}(l,x',\bar{x}')^{-1}. \end{aligned} \tag{3.13}$$

We show this identity. Substituting (3.9) to the left-hand side of (3.7) and using (3.11), one has

$$\begin{aligned} &\oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} w(l,x,\bar{x},z) w^*(l',x',\bar{x}',z) \\ &\quad = \sum_{n,n'=0}^{\infty} \big[\mathrm{e}^{\xi(x-x',(a_{l-n}^{-1}+\mathrm{e}^{-\partial_l})^{-1})}\delta_{l-n-n'l'-1}\big] w_n(l,x,\bar{x}) w_{n'}^*(l',x',\bar{x}') \\ &\quad = W(l,x,\bar{x})\,\mathrm{e}^{\xi(x-x',(a_l^{-1}+\mathrm{e}^{-\partial_l})^{-1})}\,W^*(l,x',\bar{x}')\delta_{ll'-1}. \end{aligned} \tag{3.14}$$

Similarly for the right-hand side,

$$\oint \frac{dz}{2\pi i} \bar{w}(l, x, \bar{x}, z) \bar{w}^*(l', x', \bar{x}', z)$$
$$= \sum_{n,n'=0}^{\infty} \left[e^{\xi(\bar{x}-\bar{x}', (b_{l+n}+e^{\partial_l})^{-1})} \delta_{l+n+n'l'-1} \right] \bar{w}_n(l, x, \bar{x}) \bar{w}^*_{n'}(l', x', \bar{x}')$$
$$= \bar{W}(l, x, \bar{x})\, e^{\xi(\bar{x}-\bar{x}', (b_l+e^{\partial_l})^{-1})} \bar{W}^*(l, x', \bar{x}') \delta_{ll'-1}. \tag{3.15}$$

Thus one has

$$W(l, x, \bar{x})\, e^{\xi(x-x', (a_l^{-1}+e^{-\partial_l})^{-1})} W^*(l, x', \bar{x}')$$
$$= \bar{W}(l, x, \bar{x})\, e^{\xi(\bar{x}-\bar{x}', (b_l+e^{\partial_l})^{-1})} \bar{W}^*(l, x', \bar{x}'). \tag{3.16}$$

In the case of $x = x', \bar{x} = \bar{x}'$, (3.16) reduces to

$$W(l, x, \bar{x}) W^*(l, x, \bar{x}) = \bar{W}(l, x, \bar{x}) \bar{W}^*(l, x, \bar{x}) = h(l, x, \bar{x}), \tag{3.17}$$

where we set

$$h(l, x, \bar{x}) = w_0(l, x, \bar{x}) w_0^*(l+1, x, \bar{x}) = \bar{w}_0(l, x, \bar{x}) \bar{w}_0^*(l+1, x, \bar{x}). \tag{3.18}$$

From (3.16) and (3.17) one obtains (3.13).

The exponential factors in (3.13) are expanded in terms of the multiparameter Schur functions as

$$\begin{aligned} e^{\xi(x-x', (a_l^{-1}+e^{-\partial_l})^{-1})} &= \sum_{n=0}^{\infty} p_n(x - x' | \tau^{l+1} a^{-1})\, e^{n\partial_l}, \\ e^{\xi(\bar{x}-\bar{x}', (b_l+e^{\partial_l})^{-1})} &= \sum_{n=0}^{\infty} p_n(\bar{x} - \bar{x}' | \tau^{-l} \bar{b})\, e^{-n\partial_l}. \end{aligned} \tag{3.19}$$

Thus as in the case of the Toda lattice hierarchy, we can obtain integrable systems in Sato formalism from (3.13) by using the basic properties of the multiparameter Schur functions (2.31). In what follows we focus on the difference–difference property of the multiparameter Schur functions and construct a discrete-time integrable system. Following a standard procedure we define an operator $B(l, x, \bar{x})$ by applying $D^1_{x,\lambda^{-1}}(\tau^{l+1} a^{-1})$ to the both hand sides of (3.13) and putting $x = x', \bar{x} = \bar{x}'$,

$$B(l, x, \bar{x}) = D^1_{x,\lambda^{-1}}(\tau^{l+1} a^{-1}) (W(l, x, \bar{x})\, e^{\xi(x-x', (a_l^{-1}+e^{-\partial_l})^{-1})} W(l, x', \bar{x}')^{-1})\big|_{x=x',\bar{x}=\bar{x}'}$$
$$= D^1_{x,\lambda^{-1}}(\tau^{l+1} a^{-1}) (\bar{W}(l, x, \bar{x})\, e^{\xi(\bar{x}-\bar{x}', (b_l+e^{\partial_l})^{-1})} \bar{W}(l, x', \bar{x}')^{-1})\big|_{x=x',\bar{x}=\bar{x}'}. \tag{3.20}$$

Using the basic properties (2.31), one can rewrite as

$$B = D^1_{x,\lambda^{-1}}(\tau^{l+1} a^{-1}) W \cdot W^{-1} + \frac{1}{a_{l+1}^{-1} - \lambda^{-1}} T^1_{x,\lambda}(\tau^{l+1} a^{-1}) W \cdot \left(a_{l+1}^{-1} - \lambda^{-1}\right) e^{\partial_l} W^{-1}$$
$$= D^1_{x,\lambda^{-1}}(\tau^{l+1} a^{-1}) \bar{W} \cdot \bar{W}^{-1}. \tag{3.21}$$

We here define plus and minus parts of a difference operator as follows:

$$\begin{aligned} \left(\sum_{n\in\mathbb{Z}} a_n e^{n\partial_l}\right)_{\geqslant 0} &= \sum_{n\geqslant 0} a_n e^{n\partial_l}, & \left(\sum_{n\in\mathbb{Z}} a_n e^{n\partial_l}\right)_{>0} &= \sum_{n>0} a_n e^{n\partial_l}, \\ \left(\sum_{n\in\mathbb{Z}} a_n e^{n\partial_l}\right)_{\leqslant 0} &= \sum_{n\leqslant 0} a_n e^{n\partial_l}, & \left(\sum_{n\in\mathbb{Z}} a_n e^{n\partial_l}\right)_{<0} &= \sum_{n<0} a_n e^{n\partial_l}. \end{aligned} \tag{3.22}$$

Then we note $(B)_{<0} = 0$. Thus we obtain Sato equations with respect to x,

$$\begin{aligned} &D^1_{x,\lambda^{-1}}(\tau^{l+1}a^{-1})W = BW - \frac{1}{a^{-1}_{l+1} - \lambda^{-1}} T^1_{x,\lambda^{-1}}(\tau^{l+1}a^{-1})W \cdot \left(a^{-1}_{l+1} - \lambda^{-1}\right) \mathrm{e}^{\partial_l}, \\ &D^1_{x,\lambda^{-1}}(\tau^{l+1}a^{-1})\bar{W} = B\bar{W}, \end{aligned} \tag{3.23}$$

with

$$B = \frac{1}{a^{-1}_{l+1} - \lambda^{-1}} \left(T^1_{x,\lambda^{-1}}(\tau^{l+1}a^{-1})W \cdot (1 + \left(a^{-1}_{l+1} - \lambda^{-1}\right) \mathrm{e}^{\partial_l}) W^{-1} - 1\right)_{\geqslant 0}. \tag{3.24}$$

Similarly we can obtain Sato equations with respect to $\bar{x}$,

$$\begin{aligned} &D^1_{\bar{x},\mu}(\tau^{-l}\bar{b})W = \bar{B}W, \\ &D^1_{\bar{x},\mu}(\tau^{-l}\bar{b})\bar{W} = \bar{B}\bar{W} - \frac{1}{b_{l-1} - \mu} T^1_{\bar{x},\mu}(\tau^{-l}\bar{b})\bar{W} \cdot (b_{l-1} - \mu)\, \mathrm{e}^{-\partial_l} \end{aligned} \tag{3.25}$$

with

$$\bar{B} = \frac{1}{b_{l-1} - \mu} \left(T^1_{\bar{x},\mu}(\tau^{-l}\bar{b})\bar{W} \cdot (1 + (b_{l-1} - \mu)\, \mathrm{e}^{-\partial_l})\bar{W}^{-1} - 1\right)_{\leqslant 0}. \tag{3.26}$$

The Sato equations (3.23) and (3.25) have the following equivalent expression:

$$\begin{aligned} &T^1_{x,\lambda^{-1}}(\tau^{l+1}a^{-1})W \cdot (1 + \left(a^{-1}_{l+1} - \lambda^{-1}\right) \mathrm{e}^{\partial_l}) = \left(1 + \left(a^{-1}_{l+1} - \lambda^{-1}\right)B\right) \cdot W, \\ &T^1_{x,\lambda^{-1}}(\tau^{l+1}a^{-1})\bar{W} = (1 + \left(a^{-1}_{l+1} - \lambda^{-1}\right)B) \cdot \bar{W}, \\ &T^1_{\bar{x},\mu}(\tau^{-l}\bar{b})W = (1 + (b_{l-1} - \mu)\bar{B}) \cdot W, \\ &T^1_{\bar{x},\mu}(\tau^{-l}\bar{b})\bar{W} \cdot (1 + (b_{l-1} - \mu)\, \mathrm{e}^{-\partial_l}) = (1 + (b_{l-1} - \mu)\bar{B}) \cdot \bar{W}. \end{aligned} \tag{3.27}$$

Evolution equations of the wavefunctions are given by

$$\begin{aligned} &D^1_{x,\lambda^{-1}}(\tau^{l+1}a^{-1})w = Bw, \qquad D^1_{x,\lambda^{-1}}(\tau^{l+1}a^{-1})\bar{w} = B\bar{w}, \\ &D^1_{\bar{x},\mu}(\tau^{-l}\bar{b})w = \bar{B}w, \qquad D^1_{\bar{x},\mu}(\tau^{-l}\bar{b})\bar{w} = \bar{B}\bar{w}. \end{aligned} \tag{3.28}$$

The compatibility condition leads to the Zakhalov–Shabat equation

$$\begin{aligned} &\left(1 + (b_{l-1} - \mu)T^1_{x,\lambda^{-1}}(\tau^{l+1}a^{-1})\bar{B}\right) \cdot \left(1 + \left(a^{-1}_{l+1} - \lambda^{-1}\right)B\right) \\ &\quad = \left(1 + \left(a^{-1}_{l+1} - \lambda^{-1}\right)T^1_{\bar{x},\mu}(\tau^{-l}\bar{b})B\right) \cdot (1 + (b_{l-1} - \mu)\bar{B}). \end{aligned} \tag{3.29}$$

In the homogeneous limit $a_l \to \infty, b_l \to 0$, this equation reduces to the Zakhalov–Shabat form of the discrete-time two-dimensional Toda lattice [19]. Therefore, the equation is considered as an inhomogeneous analogue of the discrete-time two-dimensional Toda lattice. See appendix A for details of the equation.

4. Lax equations and reduction to Heisenberg spin chain

In this section we construct Lax equations and show that its one-dimensional reduction gives a Bäcklund transformation of the Heisenberg spin chain.

To introduce Lax operators we need to consider shifted wavefunctions $w^+(l, x, \bar{x}, z)$ and $\bar{w}^+(l, x, \bar{x}, z)$ defined by

$$\begin{aligned} w^+(l, x, \bar{x}, z) &= w\left(l, x + \left[a^{-1}_{l+1}\right], \bar{x} + [b_{l-1}], z\right), \\ \bar{w}^+(l, x, \bar{x}, z) &= \bar{w}\left(l, x + \left[a^{-1}_{l+1}\right], \bar{x} + [b_{l-1}], z\right). \end{aligned} \tag{4.1}$$

We define shifted Sato–Wilson operators by

$$W^+(l, x, \bar{x}) = \sum_{n=0}^{\infty} w^+_n(l, x, \bar{x})\, \mathrm{e}^{-n\partial_l}, \qquad \bar{W}^+(l, x, \bar{x}) = \sum_{n=0}^{\infty} \bar{w}^+_n(l, x, \bar{x})\, \mathrm{e}^{n\partial_l}, \tag{4.2}$$

where $w_n^+(l,x,\bar{x})$ and $\bar{w}_n^+(l,x,\bar{x})$ denote the coefficients of the formal expansions

$$\begin{aligned} w^+(l,x,\bar{x},z) &= \mathrm{e}^{\xi(x,z)}\sum_{n=0}^{\infty} w_n^+(l,x,\bar{x})\frac{z^{-2}}{(z^{-1}|a^{-1})_{l+2-n}},\\ \bar{w}^+(l,x,\bar{x},z) &= \mathrm{e}^{\xi(\bar{x},z^{-1})}\sum_{n=0}^{\infty}\bar{w}_n^+(l,x,\bar{x})z(z|b)_{l-1+n}. \end{aligned} \tag{4.3}$$

The shifted wavefunctions are expressed in terms of the shifted Sato–Wilson operators as follows:

$$\begin{aligned} w^+(l,x,\bar{x},z) &= W^+(l,x,\bar{x})\left(1+a_{l+1}^{-1}\,\mathrm{e}^{\partial_l}\right)\left(1+a_l^{-1}\,\mathrm{e}^{\partial_l}\right)\frac{1}{(z^{-1}|a^{-1})_l}\,\mathrm{e}^{\xi(x,z)},\\ \bar{w}^+(l,x,\bar{x},z) &= \bar{W}^+(l,x,\bar{x})(1+b_{l-1}\,\mathrm{e}^{-\partial_l})(z|b)_l\,\mathrm{e}^{\xi(\bar{x},z^{-1})}. \end{aligned} \tag{4.4}$$

Applying the same discussion in section 3 to the bilinear identity

$$\oint\frac{\mathrm{d}z}{2\pi\mathrm{i}}w^+(l,x,\bar{x},z)w^*(l',x',\bar{x}',z) = \oint\frac{\mathrm{d}z}{2\pi\mathrm{i}}\bar{w}^+(l,x,\bar{x},z)\bar{w}^*(l',x',\bar{x}',z), \tag{4.5}$$

we can obtain the following relation among the Sato–Wilson operators:

$$\begin{aligned} &W^+(l,x,\bar{x})\left(1+a_{l+1}^{-1}\,\mathrm{e}^{\partial_l}\right)\mathrm{e}^{\xi(x-x',(a_l^{-1}+\mathrm{e}^{-\partial_l})^{-1})}W(l,x',\bar{x}')^{-1}\\ &\qquad = \bar{W}^+(l,x,\bar{x})(1+b_{l-1}\,\mathrm{e}^{-\partial_l})\,\mathrm{e}^{\xi(\bar{x}-\bar{x}',(b_l+\mathrm{e}^{\partial_l})^{-1})}\bar{W}(l,x',\bar{x}')^{-1}, \end{aligned} \tag{4.6}$$

especially,

$$W^+(l,x,\bar{x})\left(1+a_{l+1}^{-1}\,\mathrm{e}^{\partial_l}\right)W(l,x,\bar{x})^{-1} = \bar{W}^+(l,x,\bar{x})(1+b_{l-1}\,\mathrm{e}^{-\partial_l})\bar{W}(l,x,\bar{x})^{-1}. \tag{4.7}$$

Using

$$\begin{aligned} \left(1+a_{l+1}^{-1}\,\mathrm{e}^{\partial_l}\right)\mathrm{e}^{\xi(x-x',(a_l^{-1}+\mathrm{e}^{-\partial_l})^{-1})} &= \mathrm{e}^{\xi(x-x',(a_{l+1}^{-1}+\mathrm{e}^{-\partial_l})^{-1})}\left(1+a_{l+1}^{-1}\,\mathrm{e}^{\partial_l}\right),\\ (1+b_{l-1}\,\mathrm{e}^{-\partial_l})\,\mathrm{e}^{\xi(\bar{x}-\bar{x}',(b_l+\mathrm{e}^{\partial_l})^{-1})} &= \mathrm{e}^{\xi(\bar{x}-\bar{x}',(b_{l-1}+\mathrm{e}^{\partial_l})^{-1})}(1+b_{l-1}\,\mathrm{e}^{-\partial_l}) \end{aligned} \tag{4.8}$$

and (4.7), we can rewrite (4.6) as

$$\begin{aligned} &W^+(l,x,\bar{x})\,\mathrm{e}^{\xi(x-x',(a_{l+1}^{-1}+\mathrm{e}^{-\partial_l})^{-1})}W^+(l,x',\bar{x}')^{-1}\\ &\qquad = \bar{W}^+(l,x,\bar{x})\,\mathrm{e}^{\xi(\bar{x}-\bar{x}',(b_{l-1}+\mathrm{e}^{\partial_l})^{-1})}\bar{W}^+(l,x',\bar{x}')^{-1}. \end{aligned} \tag{4.9}$$

Thus we can obtain Sato equations

$$\begin{aligned} &D^1_{x,\lambda^{-1}}(\tau^{l+2}a^{-1})W^+ = B^+W^+ - \frac{1}{a_{l+2}^{-1}-\lambda^{-1}}T^1_{x,\lambda^{-1}}(\tau^{l+2}a^{-1})W^+\cdot\left(a_{l+2}^{-1}-\lambda^{-1}\right)\mathrm{e}^{\partial_l},\\ &D^1_{x,\lambda^{-1}}(\tau^{l+2}a^{-1})\bar{W}^+ = B^+\bar{W}^+,\\ &D^1_{\bar{x},\mu}(\tau^{-l+1}\bar{b})W^+ = \bar{B}^+W,\\ &D^1_{\bar{x},\mu}(\tau^{-l+1}\bar{b})\bar{W}^+ = \bar{B}^+\bar{W}^+ - \frac{1}{b_{l-2}-\mu}T^1_{\bar{x},\mu}(\tau^{-l+1}\bar{b})\bar{W}^+\cdot(b_{l-2}-\mu)\,\mathrm{e}^{-\partial_l} \end{aligned} \tag{4.10}$$

or equivalently,

$$\begin{aligned} &T^1_{x,\lambda^{-1}}(\tau^{l+2}a^{-1})W^+\cdot\left(1+\left(a_{l+2}^{-1}-\lambda^{-1}\right)\mathrm{e}^{\partial_l}\right) = \left(1+\left(a_{l+2}^{-1}-\lambda^{-1}\right)B^+\right)\cdot W^+,\\ &T^1_{x,\lambda^{-1}}(\tau^{l+2}a^{-1})\bar{W}^+ = \left(1+\left(a_{l+2}^{-1}-\lambda^{-1}\right)B^+\right)\cdot\bar{W}^+,\\ &T^1_{\bar{x},\mu}(\tau^{-l+1}\bar{b})W^+ = (1+(b_{l-2}-\mu)\bar{B}^+)\cdot W^+,\\ &T^1_{\bar{x},\mu}(\tau^{-l+1}\bar{b})\bar{W}^+\cdot(1+(b_{l-2}-\mu)\,\mathrm{e}^{-\partial_l}) = (1+(b_{l-2}-\mu)\bar{B}^+)\cdot\bar{W}^+ \end{aligned} \tag{4.11}$$

with

$$B^+ = \frac{1}{a_{l+2}^{-1} - \lambda^{-1}} \left(T^1_{x,\lambda^{-1}}(\tau^{l+2}a^{-1})W^+ \cdot (1 + \left(a_{l+2}^{-1} - \lambda^{-1}\right) e^{\partial_l})(W^+)^{-1} - 1\right)_{\geqslant 0},$$
$$\bar{B}^+ = \frac{1}{b_{l-2} - \mu} \left(T^1_{\bar{x},\mu}(\tau^{-l}\bar{b})\bar{W}^+ \cdot (1 + (b_{l-2} - \mu) e^{-\partial_l})(\bar{W}^+)^{-1} - 1\right)_{\leqslant 0}. \tag{4.12}$$

Evolution equations of the shifted wavefunctions are given by

$$D^1_{x,\lambda^{-1}}(\tau^{l+2}a^{-1})w^+ = B^+w^+, \qquad D^1_{x,\lambda^{-1}}(\tau^{l+2}a^{-1})\bar{w}^+ = B^+\bar{w}^+,$$
$$D^1_{\bar{x},\mu}(\tau^{-l+1}\bar{b})w^+ = \bar{B}^+w^+, \qquad D^1_{\bar{x},\mu}(\tau^{-l+1}\bar{b})\bar{w}^+ = \bar{B}^+\bar{w}^+. \tag{4.13}$$

We now introduce Lax operators by

$$\begin{aligned} L(l,x,\bar{x}) &= W^+(l,x,\bar{x})\, e^{\partial_l}\, W(l,x,\bar{x})^{-1},\\ \bar{L}(l,x,\bar{x}) &= \bar{W}^+(l,x,\bar{x})\, e^{-\partial_l}\, \bar{W}(l,x,\bar{x})^{-1},\\ M(l,x,\bar{x}) &= W^+(l,x,\bar{x})\left(1 + a_{l+1}^{-1} e^{\partial_l}\right)W(l,x,\bar{x})^{-1}\\ &= \bar{W}^+(l,x,\bar{x})(1 + b_{l-1} e^{-\partial_l})\bar{W}(l,x,\bar{x})^{-1}, \end{aligned} \tag{4.14}$$

where the last equality follows from (4.7). If we set

$$\begin{aligned} L(l,x,\bar{x}) &= \sum_{n=0}^{\infty} u_n(l,x,\bar{x})\, e^{(1-n)\partial_l},\\ \bar{L}(l,x,\bar{x}) &= \sum_{n=0}^{\infty} \bar{u}_n(l,x,\bar{x})\, e^{(-1+n)\partial_l}, \end{aligned} \tag{4.15}$$

then we can write

$$M(l,x,\bar{x}) = a_{l+1}^{-1}u_0(l,x,\bar{x})\, e^{\partial_l} + \bar{v}(l,x,\bar{x}) + b_{l-1}\bar{u}_0(l,x,\bar{x})\, e^{-\partial_l}. \tag{4.16}$$

By the Sato equations (3.23), (3.25) and (4.10), the Lax operators satisfy the Lax equations

$$\begin{aligned} \tilde{L} \cdot \left(1 + \left(a_{l+1}^{-1} - \lambda^{-1}\right)B\right) &= \left(1 + \left(a_{l+2}^{-1} - \lambda^{-1}\right)B^+\right) \cdot L,\\ \tilde{\bar{L}} \cdot \left(1 + \left(a_{l+1}^{-1} - \lambda^{-1}\right)B\right) &= \left(1 + \left(a_{l+2}^{-1} - \lambda^{-1}\right)B^+\right) \cdot \bar{L},\\ \tilde{M} \cdot \left(1 + \left(a_{l+1}^{-1} - \lambda^{-1}\right)B\right) &= \left(1 + \left(a_{l+2}^{-1} - \lambda^{-1}\right)B^+\right) \cdot M,\\ \hat{L} \cdot (1 + (b_{l-1} - \mu)\bar{B}) &= (1 + (b_{l-2} - \mu)\bar{B}^+) \cdot L,\\ \hat{\bar{L}} \cdot (1 + (b_{l-1} - \mu)\bar{B}) &= (1 + (b_{l-2} - \mu)\bar{B}^+) \cdot \bar{L},\\ \hat{M} \cdot (1 + (b_{l-1} - \mu)\bar{B}) &= (1 + (b_{l-2} - \mu)\bar{B}^+) \cdot M, \end{aligned} \tag{4.17}$$

where we set

$$\begin{aligned} \tilde{L} &= T^1_{x,\lambda^{-1}}(\tau^{l+2}a^{-1})W^+ \cdot e^{\partial_l} \cdot T^1_{x,\lambda^{-1}}(\tau^{l+1}a^{-1})W^{-1},\\ \tilde{\bar{L}} &= T^1_{x,\lambda^{-1}}(\tau^{l+2}a^{-1})\bar{W}^+ \cdot e^{-\partial_l} \cdot T^1_{x,\lambda^{-1}}(\tau^{l+1}a^{-1})\bar{W}^{-1},\\ \tilde{M} &= T^1_{x,\lambda^{-1}}(\tau^{l+2}a^{-1})W^+ \cdot \left(1 + a_{l+2}^{-1} e^{\partial_l}\right) \cdot T^1_{x,\lambda^{-1}}(\tau^{l+1}a^{-1})W^{-1},\\ \hat{L} &= T^1_{\bar{x},\mu}(\tau^{-l+1}\bar{b})W^+ \cdot e^{\partial_l} \cdot T^1_{\bar{x},\mu}(\tau^{-l}\bar{b})W^{-1},\\ \hat{\bar{L}} &= T^1_{\bar{x},\mu}(\tau^{-l+1}\bar{b})W^+ \cdot e^{-\partial_l} \cdot T^1_{\bar{x},\mu}(\tau^{-l}\bar{b})W^{-1},\\ \hat{M} &= T^1_{\bar{x},\mu}(\tau^{-l+1}\bar{b})W^+ \cdot (1 + b_{l-2} e^{-\partial_l}) \cdot T^1_{\bar{x},\mu}(\tau^{-l}\bar{b})W^{-1}. \end{aligned} \tag{4.18}$$

For example, the first equation in (4.17) is shown as follows:

$$\begin{aligned}\tilde{L}\cdot\left(1+\left(a_{l+1}^{-1}-\lambda^{-1}\right)B\right)\\ &= T^{1}_{x,\lambda^{-1}}(\tau^{l+2}a^{-1})W^{+}\cdot \mathrm{e}^{\partial_l}\cdot T^{1}_{x,\lambda^{-1}}(\tau^{l+1}a^{-1})W^{-1}\cdot\left(1+\left(a_{l+1}^{-1}-\lambda^{-1}\right)B\right)\\ &= T^{1}_{x,\lambda^{-1}}(\tau^{l+2}a^{-1})W^{+}\cdot \mathrm{e}^{\partial_l}\left(1+\left(a_{l+1}^{-1}-\lambda^{-1}\right)\mathrm{e}^{\partial_l}\right)W^{-1}\\ &= T^{1}_{x,\lambda^{-1}}(\tau^{l+2}a^{-1})W^{+}\cdot\left(1+\left(a_{l+2}^{-1}-\lambda^{-1}\right)\mathrm{e}^{\partial_l}\right)\mathrm{e}^{\partial_l}W^{-1}\\ &= \left(1+\left(a_{l+2}^{-1}-\lambda^{-1}\right)B^{+}\right)\cdot L.\end{aligned} \tag{4.19}$$

The wavefunctions satisfy the spectral equations

$$\begin{aligned} Lw &= zw^{+}, & w^{+} &= Mw\\ \bar{L}\bar{w} &= z^{-1}\bar{w}^{+}, & \bar{w}^{+} &= M\bar{w}.\end{aligned} \tag{4.20}$$

The Lax equations (4.17) are the compatibility condition of (3.28), (4.13) and (4.20). If we employ alternate Lax operators defined by

$$\begin{aligned}\mathcal{L}(l,x,\bar{x}) &= L(l,x,\bar{x}) + M(l,x,\bar{x})L(l,x,\bar{x})^{-1}M(l,x,\bar{x}),\\ &= W^{+}\left(\left(1+a_{l+1}^{-2}\right)\mathrm{e}^{\partial_l}+a_l^{-1}+a_{l+1}^{-1}+\mathrm{e}^{-\partial_l}\right)W^{-1},\end{aligned} \tag{4.21}$$

$$\begin{aligned}\bar{\mathcal{L}}(l,x,\bar{x}) &= \bar{L}(l,x,\bar{x}) + M(l,x,\bar{x})\bar{L}(l,x,\bar{x})^{-1}M(l,x,\bar{x}),\\ &= W^{+}\left(\left(1+b_{l-1}^{2}\right)\mathrm{e}^{-\partial_l}+b_l+b_{l-1}+\mathrm{e}^{\partial_l}\right)W^{-1},\end{aligned} \tag{4.22}$$

then they satisfy the same Lax equations

$$\begin{aligned}\tilde{\mathcal{L}}\cdot\left(1+\left(a_{l+1}^{-1}-\lambda^{-1}\right)B\right) &= \left(1+\left(a_{l+2}^{-1}-\lambda^{-1}\right)B^{+}\right)\cdot\mathcal{L},\\ \tilde{\bar{\mathcal{L}}}\cdot\left(1+\left(a_{l+1}^{-1}-\lambda^{-1}\right)B\right) &= \left(1+\left(a_{l+2}^{-1}-\lambda^{-1}\right)B^{+}\right)\cdot\bar{\mathcal{L}},\\ \tilde{M}\cdot\left(1+\left(a_{l+1}^{-1}-\lambda^{-1}\right)B\right) &= \left(1+\left(a_{l+2}^{-1}-\lambda^{-1}\right)B^{+}\right)\cdot M,\\ \hat{\mathcal{L}}\cdot(1+(b_{l-1}-\mu)\bar{B}) &= (1+(b_{l-2}-\mu)\bar{B}^{+})\cdot\mathcal{L},\\ \hat{\bar{\mathcal{L}}}\cdot(1+(b_{l-1}-\mu)\bar{B}) &= (1+(b_{l-2}-\mu)\bar{B}^{+})\cdot\bar{\mathcal{L}},\\ \hat{M}\cdot(1+(b_{l-1}-\mu)\bar{B}) &= (1+(b_{l-2}-\mu)\bar{B}^{+})\cdot M.\end{aligned} \tag{4.23}$$

The Lax equations (4.23) are the compatibility condition of (4.29), (4.13) and the spectral equations

$$\begin{aligned}\mathcal{L}w &= \zeta w^{+}, & w^{+} &= Mw,\\ \bar{\mathcal{L}}\bar{w} &= \zeta\bar{w}^{+}, & \bar{w}^{+} &= M\bar{w},\\ \zeta &= z+z^{-1}.\end{aligned} \tag{4.24}$$

We now consider a reduction to the Heisenberg spin chain. Impose the constraint

$$\begin{aligned}\mathcal{L} = \bar{\mathcal{L}} &= a_{l+1}^{-1}\alpha_{l+1}u_0\,\mathrm{e}^{\partial_l}+v+b_{l-1}\beta_{l-1}\bar{u}_0\,\mathrm{e}^{-\partial_l},\\ \alpha_l &= a_l+a_l^{-1}, \qquad \beta_l = b_l+b_l^{-1}.\end{aligned} \tag{4.25}$$

This constraint is obviously consistent with the evolutions (4.23). The spectral equations (4.24) then read

$$\left(a_{l+1}^{-1}u_0(\zeta-\alpha_{l+1})\,\mathrm{e}^{\partial_l}+\bar{v}\zeta-v+b_{l-1}\bar{u}_0(\zeta-\beta_{l-1})\,\mathrm{e}^{-\partial_l}\right)w = 0. \tag{4.26}$$

The same spectral equation is satisfied by $\bar{w}$. This spectral equation is gauge equivalent to that for the Heisenberg spin chain (B.19) under the identification $\zeta = \lambda$. Therefore, under the constraint (4.25) the Lax equations (4.23) describe a Bäcklund transformation of the Heisenberg spin chain. We show that the equation is equivalent to a discrete-time integrable system proposed by Spiridonov and Zhedanov, the R_{II}-chain [15, 16, 9]. For simplicity we consider only the evolution with respect to x. Set

$$\begin{aligned}&\mathcal{L}_1(l) = \mathcal{L}(l) = \bar{\mathcal{L}}(l) = U_1(l)\,\mathrm{e}^{\partial_l} + V_1(l) + \bar{U}_1(l)\,\mathrm{e}^{-\partial_l},\\ &\mathcal{L}_2(l) = M(l) = U_2(l)\,\mathrm{e}^{\partial_l} + V_2(l) + \bar{U}_2(l)\,\mathrm{e}^{-\partial_l}\\ &\mathcal{B}(l) = 1 + \left(a_{l+1}^{-1} - \lambda^{-1}\right)B(l) = A_1(l)\,\mathrm{e}^{\partial_l} + A_2(l),\\ &\mathcal{B}^+(l) = 1 + \left(a_{l+2}^{-1} - \lambda^{-1}\right)B^+(l) = A_1^+(l)\,\mathrm{e}^{\partial_l} + A_2^+(l),\end{aligned} \tag{4.27}$$

with

$$\begin{aligned}&U_1(l) = \alpha_{l+1}a_{l+1}^{-1}u_0(l), \qquad V_1(l) = v(l),\\ &\bar{U}_1(l) = \beta_{l-1}b_{l-1}\bar{u}_0(l),\\ &U_2(l) = a_{l+1}^{-1}u_0(l), \qquad V_2(l) = \bar{v}(l), \qquad \bar{U}_2(l) = b_{l-1}\bar{u}_0(l),\end{aligned} \tag{4.28}$$

where we omitted the dependency on the variables x and $\bar{x}$. Then the Lax equations (4.23) are written as

$$\tilde{\mathcal{L}}_i(l)\mathcal{B}(l) = \mathcal{B}^+(l)\mathcal{L}_i(l) \tag{4.29}$$

for $i = 1, 2$. Equation (4.29) read as follows:

$$\begin{aligned}&\tilde{U}_i(l) = \frac{U_i(l+1)A_1^+(l)}{A_1(l+1)}\\ &\tilde{\bar{U}}_i(l) = \frac{\bar{U}_i(l)A_2^+(l)}{A_2(l-1)},\\ &\tilde{V}_i(l) = \frac{V_i(l+1)A_1^+(l)}{A_1(l)} + \frac{U_i(l)A_2^+(l)}{A_1(l)} - \frac{U_i(l+1)A_2(l+1)A_1^+(l)}{A_1(l+1)A_1(l)},\\ &\delta_i(l+1) = \frac{A_1(l-1)A_2^+(l)}{A_1^+(l)A_2(l)}\delta_i(l),\\ &\delta_i(l) = \frac{U_i(l)A_2(l)}{A_1(l)A_1(l-1)} - \frac{V_i(l)}{A_1(l-1)} + \frac{\bar{U}_i(l)}{A_2(l-1)},\end{aligned} \tag{4.30}$$

for $i = 1, 2$. We note

$$\mathcal{B}^+(l)\delta_i(l)A_1(l-1) = A_1^+(l)\delta_i(l+1)\mathcal{B}(l), \qquad \frac{\delta_1(l)}{\delta_2(l)} = \frac{\delta_1(l+1)}{\delta_2(l+1)}. \tag{4.31}$$

Thus we have

$$\begin{aligned}\mathcal{L}_i(l) &= U_i(l)\,\mathrm{e}^{\partial_l}\frac{U_i(l)A_2(l)}{A_1(l)} + \frac{\bar{U}_i(l)A_1(l-1)}{A_2(l-1)} - A_1(l-1)\delta_i(l) + \bar{U}_i(l)\,\mathrm{e}^{-\partial_l}\\ &= \left(\frac{U_i(l)}{A_1(l)} + \frac{\bar{U}_i(l)}{A_2(l-1)}\,\mathrm{e}^{-\partial_l}\right)(A_1(l)\,\mathrm{e}^{\partial_l} + A_2(l)) - A_1(l-1)\delta_i(l)\\ &= A_1(l-1)\delta_2(l)(\mathcal{C}_i(l)\mathcal{B}(l) - \gamma_i),\end{aligned} \tag{4.32}$$

$$\begin{aligned}\tilde{\mathcal{L}}_i(l) &= \frac{U_i(l+1)A_1^+(l)}{A_1(l+1)}\,\mathrm{e}^{\partial_l} + \frac{V_i(l+1)A_1^+(l)}{A_1(l)} + \frac{U_i(l)A_2^+(l)}{A_1(l)}\\ &\quad - \frac{U_i(l+1)A_1^+(l)A_2(l+1)}{A_1(l+1)A_1(l)} + \frac{\bar{U}_i(l)A_2^+(l)}{A_2(l-1)}\,\mathrm{e}^{-\partial_l}\\ &= \left(A_1^+(l)\,\mathrm{e}^{\partial_l} + A_2^+(l)\right)\left(\frac{U_i(l)}{A_1(l)} + \frac{\bar{U}_i(l)}{A_2(l-1)}\,\mathrm{e}^{-\partial_l}\right) - A_1^+(l)\delta_i(l+1)\\ &= \mathcal{B}^+(l)\delta_2(l)A_1(l-1)\mathcal{C}_i(l) - A_1^+(l)\delta_i(l+1)\\ &= A_1^+(l)\delta_2(l+1)(\mathcal{B}(l)\mathcal{C}_i(l) - \gamma_i),\end{aligned} \tag{4.33}$$

where we set

$$\begin{aligned}
&\mathcal{C}_1(l) = \frac{1}{\delta_2(l)A_1(l-1)}\left(\frac{U_1(l)}{A_1(l)} + \frac{\bar{U}_1(l)}{A_2(l-1)}\,\mathrm{e}^{-\partial_l}\right) = \alpha_{l+1}B_1(l) + \beta_{l-1}B_2(l)\,\mathrm{e}^{-\partial_l},\\
&\mathcal{C}_2(l) = \frac{1}{\delta_2(l)A_1(l-1)}\left(\frac{U_2(l)}{A_1(l)} + \frac{\bar{U}_2(l)}{A_2(l-1)}\,\mathrm{e}^{-\partial_l}\right) = B_1(l) + B_2(l)\,\mathrm{e}^{-\partial_l},\\
&B_1(l) = \frac{U_2(l)}{\delta_2(l)A_1(l)A_1(l-1)}, \qquad B_2(l) = \frac{\bar{U}_2(l)}{\delta_2(l)A_1(l-1)A_2(l-1)},\\
&\gamma_1 = \frac{\delta_1(l)}{\delta_2(l)}, \qquad \gamma_2 = 1.
\end{aligned} \tag{4.34}$$

The compatibility condition leads to

$$\tilde{A}_1(l-1)\tilde{\delta}_2(l)(\tilde{\mathcal{C}}_i(l)\tilde{\mathcal{B}}(l) - \tilde{\gamma}_i) = A_1^{+}(l)\delta_2(l+1)(\mathcal{B}(l)\mathcal{C}_i(l) - \gamma_i) \tag{4.35}$$

for $i = 1, 2$, explicitly,

$$\begin{aligned}
&\frac{\tilde{\alpha}_{l+1}\tilde{A}_2(l)\tilde{B}_1(l) + \tilde{\beta}_{l-1}\tilde{A}_1(l-1)\tilde{B}_2(l) - \tilde{\gamma}_1}{\tilde{A}_1(l)\tilde{B}_1(l)}\\
&\qquad = \frac{\alpha_{l+1}A_2(l)B_1(l) + \beta_l A_1(l)B_2(l+1) - \gamma_1}{\tilde{A}_1(l)\tilde{B}_1(l+1)},\\
&\frac{\tilde{A}_2(l)\tilde{B}_1(l) + \tilde{A}_1(l-1)\tilde{B}_2(l) - 1}{\tilde{A}_2(l)\tilde{B}_1(l)} = \frac{A_2(l)B_1(l) + A_1(l)B_2(l+1) - 1}{A_1(l)B_1(l+1)},\\
&\qquad \frac{\tilde{A}_2(l-1)\tilde{B}_2(l)}{\tilde{A}_1(l)\tilde{B}_1(l)} = \frac{A_2(l)B_2(l)}{A_1(l)B_1(l+1)}, \qquad \tilde{\alpha}_l = \alpha_{l+1}, \qquad \tilde{\beta}_l = \beta_l.
\end{aligned} \tag{4.36}$$

This discrete-time integrable system is the so-called R_{II}-chain.

The degeneration of the Heisenberg spin chain is pointed out by Sklyanin [14]. Zhedanov [22] also discussed such a degeneration from the viewpoint of a generalized eigenvalue problem. We show that the degeneration is naturally understood as a homogeneous limit of the model. In the limit $a_l \to \infty$, (4.26) reduces to

$$(-u_0\,\mathrm{e}^{\partial_l} + \bar{v}\zeta - v + b_{l-1}\bar{u}_0(\zeta - \beta_{l-1})\,\mathrm{e}^{-\partial_l})w = 0, \tag{4.37}$$

which is the spectral equation for the discrete self-trapping chain [5] or the R_I chain [8, 21]. In the case of $b_l = \text{const}$, equation (4.37) is equivalent to the spectral equation for the relativistic Toda lattice [10, 12, 17]. In the limit $b_l \to 0$, (4.26) reduces to

$$\left(a_{l+1}^{-1}u_0(\zeta - \alpha_{l+1})\,\mathrm{e}^{\partial_l} + \bar{v}\zeta - v - \bar{u}_0\,\mathrm{e}^{-\partial_l}\right)w = 0, \tag{4.38}$$

which is the dual equation of (4.37). Equations (4.37) and (4.38) further reduce to

$$(-u_0\,\mathrm{e}^{\partial_l} + \bar{v}\zeta - v - \bar{u}_0\,\mathrm{e}^{-\partial_l})w = 0, \tag{4.39}$$

in the limit $b_l \to 0$ and $a_l \to \infty$, respectively. Equation (4.39) is the celebrated spectral equation for the Toda lattice.

5. Concluding remarks

In this paper we have introduced a τ-function on the inhomogeneous lattice and constructed an associated discrete-time integrable system in Sato and Lax formalisms. We have shown that a one-dimensional reduction of the system gives a Bäcklund transformation of the Heisenberg spin chain, which is equivalent to the R_{II}-chain proposed by Spiridonov and Zhedanov.

In our approach the τ-function was introduced by an inhomogeneous deformation of that of the Toda lattice hierarchy. Therefore, the Heisenberg spin chain can be regarded as an inhomogeneous deformation of the Toda lattice. The discrete self-trapping chain is located between the Heisenberg spin chain and the Toda lattice. This perspective provides a natural explanation to the degeneration of the Heisenberg spin chain pointed out by Sklyanin.

The important point of our discussion is that the multiparameter Schur function played a key role. The function is a kind of inhomogeneous analogue of the Schur functions. To clarify a connection between various kinds of Schur functions and inhomogeneous integrable systems is a future subject to be studied.

Acknowledgments

The author would like to thank Professor Yuji Kodama and Dr Virgil Pierce for valuable discussions. This work was partly supported by the Grant-in-Aid for JSPS Fellows, The Ministry of Education, Culture, Sports, Science and Technology, Japan.

Appendix A. Approach based on Hirota formalism

In this section we present an approach based on Hirota formalism.

As in the case of the KP hierarchy [6] or the Toda lattice hierarchy [20], the τ-function (3.1) satisfies the Fay-type identities

$$\begin{aligned}
&\tau\big(l, x-\big[z_1^{-1}\big], \bar{x}\big)\tau\big(l, x-\big[z_2^{-1}\big], \bar{x}-[z_3]+[z_4]\big)\\
&\qquad -\tau\big(l, x-\big[z_2^{-1}\big], \bar{x}\big)\tau\big(l, x-\big[z_1^{-1}\big], \bar{x}-[z_3]+[z_4]\big)\\
&\quad=\big(z_1^{-1}-z_2^{-1}\big)(z_3-z_4)\tau\big(l+1, x-\big[a_l^{-1}\big], \bar{x}+[b_l]-[z_3]\big)\\
&\qquad\times\tau\big(l-1, x+\big[a_{l-1}^{-1}\big]-\big[z_1^{-1}\big]-\big[z_2^{-1}\big], \bar{x}-[b_{l-1}]+[z_4]\big),
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
&\tau\big(l, x-\big[z_1^{-1}\big]+\big[z_2^{-1}\big], \bar{x}-[z_4]\big)\tau(l, x, \bar{x}-[z_3])\\
&\qquad -\tau\big(l, x-\big[z_1^{-1}\big]+\big[z_2^{-1}\big], \bar{x}-[z_3]\big)\tau(l, x, \bar{x}-[z_4])\\
&\quad=\big(z_1^{-1}-z_2^{-1}\big)(z_3-z_4)\tau\big(l+1, x-\big[a_l^{-1}\big]+\big[z_2^{-1}\big], \bar{x}+[b_l]-[z_3]-[z_4]\big)\\
&\qquad\times\tau\big(l-1, x+\big[a_{l-1}^{-1}\big]-\big[z_1^{-1}\big], \bar{x}-[b_{l-1}]\big),
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
&\big(z_1^{-1}-z_3^{-1}\big)\tau\big(l+1, x-\big[a_l^{-1}\big]-\big[z_2^{-1}\big]+\big[z_3^{-1}\big], \bar{x}+[b_l]-[z_4]\big)\tau\big(l, x-\big[z_1^{-1}\big], \bar{x}\big)\\
&\qquad -\big(z_2^{-1}-z_3^{-1}\big)\tau\big(l+1, x-\big[a_l^{-1}\big]-\big[z_1^{-1}\big]+\big[z_3^{-1}\big], \bar{x}+[b_l]-[z_4]\big)\\
&\qquad\times\tau\big(l, x-\big[z_2^{-1}\big], \bar{x}\big)\\
&\quad=\big(z_1^{-1}-z_2^{-1}\big)\tau\big(l+1, x-\big[a_l^{-1}\big], \bar{x}+[b_l]-[z_4]\big)\\
&\qquad\times\tau\big(l, x-\big[z_1^{-1}\big]-\big[z_2^{-1}\big]+\big[z_3^{-1}\big], \bar{x}\big),
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
&(z_2-z_4)\tau\big(l+1, x-\big[a_l^{-1}\big], \bar{x}+[b_l]-[z_2]\big)\tau\big(l, x-\big[z_1^{-1}\big], \bar{x}-[z_3]+[z_4]\big)\\
&\qquad -(z_3-z_4)\tau\big(l+1, x-\big[a_l^{-1}\big], \bar{x}+[b_l]-[z_3]\big)\tau\big(l, x-\big[z_1^{-1}\big], \bar{x}-[z_2]+[z_4]\big)\\
&\quad=(z_2-z_3)\tau\big(l+1, x-\big[a_l^{-1}\big], \bar{x}+[b_l]-[z_2]-[z_3]+[z_4]\big)\\
&\qquad\times\tau\big(l, x-\big[z_1^{-1}\big], \bar{x}\big).
\end{aligned} \tag{A.4}$$

We show that these identities follow from the bilinear identity (3.3). Replacing x and x' with $x - [a_l^{-1}]$ and $x' + [a_{l'-1}^{-1}]$ respectively, one has

$$\oint \frac{dz}{2\pi i} \frac{(z^{-1}|a^{-1})_{l'}}{(z^{-1}|a^{-1})_l} e^{\xi(x-x',z)} \tau(l, x - [z^{-1}], \bar{x}) \tau(l', x' + [z^{-1}], \bar{x}')$$
$$= \oint \frac{dz}{2\pi i} \frac{(z|b)_l}{(z|b)_{l'}} e^{\xi(\bar{x}-\bar{x}', z^{-1})} \tau\left(l+1, x - [a_l^{-1}], \bar{x} + [b_l] - [z]\right)$$
$$\times \tau\left(l'-1, x' + [a_{l'-1}^{-1}], \bar{x}' - [b_{l'-1}] + [z]\right). \tag{A.5}$$

If l', x' and $\bar{x}'$ are specialized as $l = l'$, $x - x' = [z_1^{-1}] + [z_2^{-1}]$ and $\bar{x} - \bar{x}' = [z_3] - [z_4]$,

$$\oint \frac{dz}{2\pi i} \frac{1}{(1 - z_1^{-1}z)(1 - z_2^{-1}z)} \tau(l, x - [z^{-1}], \bar{x})$$
$$\times \tau\left(l, x - [z_1^{-1}] - [z_2^{-1}] + [z^{-1}], \bar{x} - [z_3] + [z_4]\right)$$
$$= \oint \frac{dz}{2\pi i} \frac{z - z_4}{z - z_3} \tau\left(l+1, x - [a_l^{-1}], \bar{x} + [b_l] - [z]\right)$$
$$\times \tau\left(l-1, x + [a_{l-1}^{-1}] - [z_1^{-1}] - [z_2^{-1}], \bar{x} - [b_{l-1}] - [z_3] + [z_4] + [z]\right), \tag{A.6}$$

which leads to the identity (A.1). The rest identities (A.2)–(A.4) can be obtained in a similar manner.

Let $\lambda = (\lambda_l)_{l\in\mathbb{Z}}$ and $\mu = (\mu_l)_{l\in\mathbb{Z}}$ be sets mutually distinct in $\mathbb{C}$. Following [6] we introduce the discrete variables $m, n, \bar{m}, \bar{n}$ as

$$\tau(l, m, n, \bar{m}, \bar{n}) = \tau\left(l, x + \sum_{i=0}^{m+n-1} [a_{l+i}^{-1}] - \sum_{i=0}^{n-1} [\lambda_i^{-1}], \bar{x} + \sum_{i=0}^{\bar{m}+\bar{n}-1} [b_{l-i-1}] - \sum_{i=0}^{\bar{n}-1} [\mu_{-i}]\right), \tag{A.7}$$

where the following notation is used:

$$\sum_{i=0}^{n} a_i = \begin{cases} \sum_{i=0}^{n} a_i & (n > -1), \\ 0 & (n = -1), \\ -\sum_{i=1}^{-n-1} a_{-i} & (n < -1). \end{cases} \tag{A.8}$$

Then we can obtain Hirota equations from the Fay-type identities (A.1)–(A.4). For example, (A.1) yields the Hirota equation

$$\tau(l, m, n+1, \bar{m}, \bar{n}+1)\tau(l, m, n, \bar{m}, \bar{n}) - \tau(l, m, n+1, \bar{m}, \bar{n})\tau(l, m, n, \bar{m}, \bar{n}+1)$$
$$= \left(a_{l+m+n}^{-1} - \lambda_n^{-1}\right)(\mu_{-\bar{n}} - b_{l-\bar{m}-\bar{n}-1})\tau(l+1, m, n, \bar{m}, \bar{n}+1)$$
$$\times \tau(l-1, m, n+1, \bar{m}, \bar{n}). \tag{A.9}$$

Similarly we have

$$\tau(l, m+1, n, \bar{m}, \bar{n}+1)\tau(l, m, n, \bar{m}, \bar{n}) - \tau(l, m+1, n, \bar{m}, \bar{n})\tau(l, m, n, \bar{m}, \bar{n}+1)$$
$$= a_{l+m+n}^{-1}(\mu_{-\bar{n}} - b_{l-\bar{m}-\bar{n}-1})\tau(l+1, m, n, \bar{m}, \bar{n}+1)$$
$$\times \tau(l-1, m+1, n, \bar{m}, \bar{n}), \tag{A.10}$$

$$a_{l+m+n+1}^{-1}\tau(l+1,m,n+1,\bar m,\bar n)\tau(l,m+1,n,\bar m,\bar n) - \lambda_n^{-1}\tau(l+1,m,n,\bar m,\bar n)\tau(l,m+1,n+1,\bar m,\bar n) = \left(a_{l+m+n+1}^{-1}-\lambda_n^{-1}\right)\tau(l+1,m+1,n,\bar m,\bar n)\tau(l,m,n+1,\bar m,\bar n), \tag{A.11}$$

and so on. We note that all the Hirota equations (A.9)–(A.11) are Bäcklund transformations of the discrete-time two-dimensional Toda lattice [2].

We now construct the Sato equations (3.23) and (3.25) by using the Fay-type identities. Set

$$\begin{aligned} w(l,m,n,\bar m,\bar n,z) &= w\left(l, x+\sum_{i=0}^{m+n-1}\left[a_{l+i+1}^{-1}\right]-\sum_{i=0}^{n-1}\left[\lambda_i^{-1}\right], \bar x+\sum_{i=0}^{\bar m+\bar n-1}[b_{l-i-1}]-\sum_{i=0}^{\bar n-1}[\mu_{-i}]\right),\\ \bar w(l,m,n,\bar m,\bar n,z) &= \bar w\left(l, x+\sum_{i=0}^{m+n-1}\left[a_{l+i+1}^{-1}\right]-\sum_{i=0}^{n-1}\left[\lambda_i^{-1}\right], \bar x+\sum_{i=0}^{\bar m+\bar n-1}[b_{l-i-1}]-\sum_{i=0}^{\bar n-1}[\mu_{-i}]\right). \end{aligned} \tag{A.12}$$

Then the Fay-type identities (A.1)–(A.4) yield the recurrence relations

$$\begin{aligned} w(l,m,n+1,\bar m,\bar n,z) &= \left(a_{l+m+n+1}^{-1}-\lambda_n^{-1}\right)\frac{\tau(l+1,m+1,n,\bar m,\bar n+1)\tau(l,m,n+1,\bar m,\bar n)}{\tau(l+1,m,n,\bar m,\bar n+1)\tau(l,m+1,n+1,\bar m,\bar n)}\\ &\quad\times w(l+1,m,n,\bar m,\bar n+1,z)\\ &\quad+\frac{\tau(l+1,m,n+1,\bar m,\bar n+1)\tau(l,m+1,n,\bar m,\bar n)}{\tau(l+1,m,n,\bar m,\bar n+1)\tau(l,m+1,n+1,\bar m,\bar n)}w(l,m,n,\bar m,\bar n,z), \end{aligned} \tag{A.13}$$

$$\begin{aligned} w(l,m,n,\bar m,\bar n+1,z) &= \frac{\tau(l,m,n+1,\bar m,\bar n+1)\tau(l,m+1,n,\bar m,\bar n)}{\tau(l,m+1,n,\bar m,\bar n+1)\tau(l,m,n+1,\bar m,\bar n)}w(l,m,n,\bar m,\bar n,z)\\ &\quad+(b_{l-\bar m-\bar n-1}-\mu_{-\bar n})\frac{\tau(l+1,m,n,\bar m,\bar n+1)\tau(l-1,m+1,n+1,\bar m,\bar n)}{\tau(l,m+1,n,\bar m,\bar n+1)\tau(l,m,n+1,\bar m,\bar n)}\\ &\quad\times w(l-1,m,n+1,\bar m,\bar n,z). \end{aligned} \tag{A.14}$$

The same recurrence relations are satisfied by $\bar w(l,m,n,\bar m,\bar n,z)$. If $\mu_{-\bar n}$ is specialized as $\mu_{-\bar n}=b_{l-\bar m-\bar n}$, (A.13) reduces to

$$\begin{aligned} w(l,m,n+1,\bar m,\bar n,z) &= \left(a_{l+m+n+1}^{-1}-\lambda_n^{-1}\right)\frac{\tau(l+1,m+1,n,\bar m,\bar n)\tau(l,m,n+1,\bar m,\bar n)}{\tau(l+1,m,n,\bar m,\bar n)\tau(l,m+1,n+1,\bar m,\bar n)}\\ &\quad\times w(l+1,m,n,\bar m,\bar n,z)\\ &\quad+\frac{\tau(l+1,m,n+1,\bar m,\bar n)\tau(l,m+1,n,\bar m,\bar n)}{\tau(l+1,m,n,\bar m,\bar n)\tau(l,m+1,n+1,\bar m,\bar n)}w(l,m,n,\bar m,\bar n,z). \end{aligned} \tag{A.15}$$

Using the Hirota equations (A.10), we can rewrite as

$$w(l,m,n+1,\bar m,\bar n,z) = \left(1+\left(a_{l+m+n+1}^{-1}-\lambda_n^{-1}\right)B(l,m,n,\bar m,\bar n)\right)w(l,m,n,\bar m,\bar n,z) \tag{A.16}$$

with

$$B(l,m,n,\bar{m},\bar{n}) = E(l,m,n,\bar{m},\bar{n})\,\mathrm{e}^{\partial_l} + a_{l+m+n+1}(E(l,m,n,\bar{m},\bar{n}) - 1),$$
$$E(l,m,n,\bar{m},\bar{n}) = \frac{\tau(l+1,m+1,n,\bar{m},\bar{n})\tau(l,m,n+1,\bar{m},\bar{n})}{\tau(l+1,m,n,\bar{m},\bar{n})\tau(l,m+1,n+1,\bar{m},\bar{n})}. \tag{A.17}$$

This implies the Sato equation (3.23). Similarly from (A.14) and (A.11), we can obtain

$$w(l,m,n,\bar{m},\bar{n}+1,z) = (1 + (b_{l-\bar{m}-\bar{n}-1} - \mu_{-\bar{n}})\bar{B}(l,m,n,\bar{m},\bar{n}))w(l,m,n,\bar{m},\bar{n},z) \tag{A.18}$$

with

$$\bar{B}(l,m,n,\bar{m},\bar{n}) = a_{l+m+n}^{-1}F(l,m,n,\bar{m},\bar{n}) + F(l,m,n,\bar{m},\bar{n})\,\mathrm{e}^{-\partial_l},$$
$$F(l,m,n,\bar{m},\bar{n}) = \frac{\tau(l+1,m,n,\bar{m},\bar{n}+1)\tau(l-1,m+1,n,\bar{m},\bar{n})}{\tau(l,m+1,n,\bar{m},\bar{n}+1)\tau(l,m,n,\bar{m},\bar{n})}, \tag{A.19}$$

which implies the Sato equation (3.25). The remaining Sato equations can be obtained in a similar manner. The compatibility condition of (A.16) and (A.18) leads to the Zakhalov–Shabat equation

$$\begin{aligned}
&((\lambda_n - a_{l+m+n})E(l-1,m,n,\bar{m},\bar{n}) + a_{l+m+n})F(l,m,n+1,\bar{m},\bar{n})\\
&\quad = ((\lambda_n - a_{l+m+n+1})E(l,m,n,\bar{m},\bar{n}+1) + a_{l+m+n+1})\\
&\qquad \times F(l,m,n,\bar{m},\bar{n}),\\
&E(l,m,n,\bar{m},\bar{n})\big(1 + a_{l+m+n}^{-1}(b_{l-\bar{m}-\bar{n}-1} - \mu_{-\bar{n}})F(l,m,n+1,\bar{m},\bar{n})\big)\\
&\quad = E(l,m,n,\bar{m},\bar{n}+1)\\
&\qquad \times \big(1 + a_{l+m+n+1}^{-1}(b_{l-\bar{m}-\bar{n}-1} - \mu_{-\bar{n}})F(l+1,m,n,\bar{m},\bar{n})\big),
\end{aligned} \tag{A.20}$$

which is considered as an inhomogeneous analogue of the discrete-time two-dimensional Toda lattice.

A specialization of the recurrence relations (A.13) and (A.14) also gives an explicit expression of one of the Lax operators (4.14). If λ_n is specialized as $\lambda_n = \infty$, (A.13) reduces to

$$\begin{aligned}
&w(l,m+1,n,\bar{m},\bar{n},z)\\
&\quad = a_{l+m+n+1}^{-1}\frac{\tau(l+1,m+1,n,\bar{m},\bar{n})\tau(l,m+1,n,\bar{m},\bar{n})}{\tau(l+1,m,n,\bar{m},\bar{n})\tau(l,m+2,n,\bar{m},\bar{n})}\\
&\qquad \times w(l+1,m,n,\bar{m},\bar{n}+1,z)\\
&\qquad + \frac{\tau(l+1,m+1,n,\bar{m},\bar{n})\tau(l,m+1,n,\bar{m},\bar{n})}{\tau(l+1,m,n,\bar{m},\bar{n})\tau(l,m+2,n,\bar{m},\bar{n})}w(l,m,n,\bar{m},\bar{n},z).
\end{aligned} \tag{A.21}$$

Similarly (A.14) reduces to

$$\begin{aligned}
&w(l,m,n,\bar{m}+1,\bar{n},z)\\
&\quad = \frac{\tau(l,m+1,n,\bar{m},\bar{n})\tau(l,m,n,\bar{m}+1,\bar{n})}{\tau(l,m+1,n,\bar{m}+1,\bar{n})\tau(l,m,n,\bar{m},\bar{n})}w(l,m,n,\bar{m},\bar{n},z)\\
&\qquad + b_{l-\bar{m}-\bar{n}-1}\frac{\tau(l+1,m,n,\bar{m}+1,\bar{n})\tau(l-1,m+1,n,\bar{m},\bar{n})}{\tau(l,m+1,n,\bar{m}+1,\bar{n})\tau(l,m,n,\bar{m},\bar{n})}\\
&\qquad \times w(l-1,m,n,\bar{m},\bar{n},z).\\
&w(l,m+1,n,\bar{m}+1,\bar{n},z)\\
&\quad = \frac{\tau(l,m+2,n,\bar{m},\bar{n})\tau(l,m+1,n,\bar{m}+1,\bar{n})}{\tau(l,m+2,n,\bar{m}+1,\bar{n})\tau(l,m+1,n,\bar{m},\bar{n})}w(l,m+1,n,\bar{m},\bar{n},z)
\end{aligned} \tag{A.22}$$

From (A.21) and (A.22), we obtain

$$w^{+}(l,m,n,\bar{m},\bar{n},z)=w(l,m+1,n,\bar{m}+1,\bar{n},z)$$
$$=M(l,m,n,\bar{m},\bar{n})w(l,m,n,\bar{m},\bar{n},z) \qquad \text{(A.23)}$$

with

$$\begin{aligned} M(l,m,n,\bar{m},\bar{n}) &= a_{l+m+n+1}^{-1}u_0(l,m,n,\bar{m},\bar{n})\,\mathrm{e}^{\partial_l}+u_0(l,m,n,\bar{m},\bar{n}) \\ &\quad +a_{l+m+n}^{-1}b_{l-\bar{m}-\bar{n}-1}\bar{u}_0(l,m,n,\bar{m},\bar{n})+b_{l-\bar{m}-\bar{n}-1}\bar{u}_0(l,m,n,\bar{m},\bar{n})\,\mathrm{e}^{-\partial_l}, \\ u_0(l,m,n,\bar{m},\bar{n}) &= \frac{\tau(l+1,m+1,n,\bar{m},\bar{n})\tau(l,m+1,n,\bar{m}+1,\bar{n})}{\tau(l+1,m,n,\bar{m},\bar{n})\tau(l,m+2,n,\bar{m}+1,\bar{n})}, \\ \bar{u}_0(l,m,n,\bar{m},\bar{n}) &= \frac{\tau(l+1,m+1,n,\bar{m}+1,\bar{n})\tau(l-1,m+1,n,\bar{m},\bar{n})}{\tau(l,m+2,n,\bar{m}+1,\bar{n})\tau(l,m,n,\bar{m},\bar{n})}, \end{aligned} \qquad \text{(A.24)}$$

where u_0 and $\bar{u}_0$ are the leading terms of the Lax operators L and $\bar{L}$, respectively.

Appendix B. *L*-operators for Heisenberg spin chain

In this section we give a brief description of the Heisenberg spin chain (cf [1, 14]) and its L-operators.

The model is described in terms of spin variables with length s_n,

$$\vec{S}_n=(S_{1n}\quad S_{2n}\quad S_{3n})^{\top},\qquad \vec{S}_n\cdot\vec{S}_n=s_n^2, \qquad \text{(B.1)}$$

having the Poisson brackets

$$\{S_{im},S_{jn}\}=-\sqrt{-1}\sum_{k=1}^{3}\varepsilon_{ijk}S_{kn}\delta_{mn}, \qquad \text{(B.2)}$$

where ε_{ijk} is the skew-symmetric tensor. Introduce the L-operator as

$$L_n(\lambda)=(\lambda-c_n)I+\sum_{i=1}^{3}S_{in}\sigma_i=\begin{pmatrix}\lambda-c_n+S_{3n} & S_{1n}-\sqrt{-1}S_{2n}\\ S_{1n}+\sqrt{-1}S_{2n} & \lambda-c_n-S_{3n}\end{pmatrix}, \qquad \text{(B.3)}$$

where λ is a spectral parameter, c_n is an arbitrary parameter and $\sigma_i\,(i=1,2,3)$ are the Pauli matrices. We note that the model is an inhomogeneous model, which means that the model contains two inhomogeneous parameters, the spin length s_n and the shift c_n which depend on the lattice site n.

We impose the N-periodic boundary condition

$$\vec{S}_{n+N}=\vec{S}_n,\qquad c_{n+N}=c_n. \qquad \text{(B.4)}$$

The transition matrix is defined by a product of the L-operators

$$T(\lambda)=L_N(\lambda)\cdots L_1(\lambda). \qquad \text{(B.5)}$$

The spectral invariants of the model are given by the coefficients of the characteristic polynomial

$$W(\lambda,\mu)=\det(\mu I-T(\lambda))=\mu^2-\operatorname{tr}T(\lambda)\mu+\det T(\lambda). \qquad \text{(B.6)}$$

Since

$$\det T(\lambda)=\prod_{i=1}^{N}(\lambda-s_i-c_i)(\lambda+s_i-c_i), \qquad \text{(B.7)}$$

only $\operatorname{tr}T(\lambda)$ is a nontrivial spectral invariant:

$$t(\lambda)=\operatorname{tr}T(\lambda). \qquad \text{(B.8)}$$

The Hamiltonians of the model are obtained by the expansion of $t(\lambda)$.

Integrability of the model can be shown in a standard procedure. The L-operator satisfies the fundamental Poisson bracket

$$\{L_m(\lambda) \overset{\otimes}{,} L_n(\mu)\} = [r(\lambda-\mu), L_m(\lambda)\otimes L_n(\mu)]\delta_{mn} \tag{B.9}$$

for the r-matrix

$$r(\lambda) = -\frac{1}{\lambda}P_{12}, \tag{B.10}$$

where P_{12} is the permutation operator in $\mathbb{C}^2\otimes\mathbb{C}^2$. Using the Poisson bracket recursively, one can show the Poisson bracket for the transition matrix

$$\{T(\lambda) \overset{\otimes}{,} T(\mu)\} = [r(\lambda-\mu), T(\lambda)\otimes T(\mu)], \tag{B.11}$$

from which the commutativity of the Hamiltonians follows:

$$\{t(\lambda), t(\mu)\} = 0. \tag{B.12}$$

The L-operator is a 2×2 Lax operator. In the case of the Toda lattice [14], there exists an $N\times N$ Lax operator which is dual to the L-operator in the sense that the corresponding characteristic polynomials are equivalent. We next introduce a pair of $N\times N$ Lax operators dual to the L-operators (B.3). Let (q_n, p_n) be the canonical variables having the Poisson brackets

$$\{q_m, q_n\} = \{p_m, p_n\} = 0, \qquad \{p_m, q_n\} = \delta_{mn}. \tag{B.13}$$

The spin variables can be parametrized in terms of the variables as

$$\vec{S}_n = \left(\tfrac{1}{2}p_n\left(1-q_n^2\right)+s_nq_n \quad \tfrac{1}{2\sqrt{-1}}p_n\left(1+q_n^2\right)-\tfrac{1}{\sqrt{-1}}s_nq_n \quad q_np_n-s_n\right)^{\top}. \tag{B.14}$$

The L-operator is written in the variables as

$$\begin{aligned} L_n(\lambda) &= \begin{pmatrix} \lambda-\alpha_{n+1}+q_np_n & q_n(\alpha_{n+1}-\beta_n-q_np_n) \\ p_n & \lambda-\beta_n-q_np_n \end{pmatrix}, \\ \alpha_{n+1} &= s_n+c_n, \qquad \beta_n = -s_n+c_n. \end{aligned} \tag{B.15}$$

Consider the associated spectral equation

$$\begin{pmatrix} \varphi_{n+1}(\lambda) \\ \psi_{n+1}(\lambda) \end{pmatrix} = L_n(\lambda)\begin{pmatrix} \varphi_n(\lambda) \\ \psi_n(\lambda) \end{pmatrix}. \tag{B.16}$$

Eliminating $\varphi_n(\lambda)$ we have a three-term recurrence relation for $\psi_n(\lambda)$,

$$\begin{aligned} &\psi_{n+1}(\lambda) - ((1+v_n)\lambda-u_n)\psi_n(\lambda) + v_n(\lambda-\alpha_n)(\lambda-\beta_{n-1})\psi_{n-1}(\lambda) = 0, \\ &u_n = p_n(q_n-q_{n+1}) + \beta_n + \alpha_{n+1}\frac{p_n}{p_{n-1}}, \qquad v_n = \frac{p_n}{p_{n-1}}. \end{aligned} \tag{B.17}$$

Following [22] apply the variable transformation

$$\psi_n(\lambda) = \tilde{\psi}_n(\lambda)\prod_{i=1}^{n}(\lambda-\alpha_i). \tag{B.18}$$

Then the relation is transformed to

$$(\lambda-\alpha_{n+1})\tilde{\psi}_{n+1}(\lambda) - ((1+v_n)\lambda-u_n)\tilde{\psi}_n(\lambda) + v_n(\lambda-\beta_{n-1})\tilde{\psi}_{n-1}(\lambda) = 0. \tag{B.19}$$

We assume the quasi-periodicity of $\tilde{\psi}_n(\lambda)$,

$$\tilde{\psi}_{n+N}(\lambda) = \mu\tilde{\psi}_n(\lambda). \tag{B.20}$$

Then (B.19) turns into a generalized eigenvalue problem

$$\begin{aligned}&\mathcal{L}_1(\mu)\Psi(\lambda) = \lambda\mathcal{L}_2(\mu)\Psi(\lambda),\\&\Psi(\lambda) = (\tilde{\psi}_1(\lambda) \quad \tilde{\psi}_2(\lambda) \quad \cdots \quad \tilde{\psi}_N(\lambda))^\top,\end{aligned} \tag{B.21}$$

where $\mathcal{L}_1(\mu)$ and $\mathcal{L}_2(\mu)$ are $N \times N$ Lax operators

$$\begin{aligned}\mathcal{L}_1(\mu) &= \begin{pmatrix} -u_1 & \alpha_2 & & & \mu^{-1}\beta_N v_1 \\ \beta_1 v_2 & -u_2 & \alpha_3 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{N-2}v_{N-1} & -u_{N-1} & \alpha_N \\ \mu\alpha_1 & & & \beta_{N-1}v_N & -u_N \end{pmatrix},\\ \mathcal{L}_2(\mu) &= \begin{pmatrix} -1-v_1 & 1 & & & \mu^{-1}v_1 \\ v_2 & -1-v_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & v_{N-1} & -1-v_{N-1} & 1 \\ \mu & & & v_N & -1-v_N \end{pmatrix}.\end{aligned} \tag{B.22}$$

Let $\tilde{W}(\lambda, \mu)$ be a modified characteristic polynomial of the transition matrix $T(\lambda)$,

$$\begin{aligned}\tilde{W}(\lambda, \mu) &= \det\left(\mu I - \frac{1}{\prod_{i=1}^N(\lambda - \alpha_i)}T(\lambda)\right)\\ &= \prod_{i=1}^N(\lambda - \alpha_i)\mu^2 - t(\lambda)\mu + \prod_{i=1}^N(\lambda - \beta_i).\end{aligned} \tag{B.23}$$

Then we can show a duality between $T(\lambda)$ and $\mathcal{L}_1(\mu)$, $\mathcal{L}_2(\mu)$,

$$\tilde{W}(\lambda, \mu) = \det\left(\mu I - \frac{1}{\prod_{i=1}^N(\lambda - \alpha_i)}T(\lambda)\right) = \mu\det(\lambda\mathcal{L}_2(\mu) - \mathcal{L}_1(\mu)). \tag{B.24}$$

This duality enables us to deal with the generalized eigenvalue problem (B.21) instead of the original spectral problem (B.16).

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